



# MT365

## Audio Notes 1

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CDA5674 PP 4 - 33

CDA 5675 PP 36 - end

This booklet contains the printed material for use with Audio-tapes 1 and 2, which go with Blocks 1 and 2.

You will need to play the tape at the same time as you study the frames on the following pages.

Place the cassette player within easy reach. There are points on the tape where we have indicated that we want you to stop the tape and do some work for yourself, but you will probably find it necessary to stop the tape more often than this. Indeed, you should get into the habit of frequently stopping the tape and giving yourself time to think.

Make sure that you have paper and pencil handy before starting each tape sequence.

## BLOCK 1

### Notes for Networks 1

There are two sequences on the tape. The heading numbers 3.3 and 4.1 below refer to the corresponding sections in *Networks 1*.

In each sequence we describe an algorithm for solving a particular network flow problem.

We introduce the algorithm by considering an example, and we then ask you to solve a problem for yourself. You may find it helpful to listen to the discussion of the example more than once before tackling the problem. If you still have difficulty with the problem, listen to the discussion of the solution on the tape, and then try to solve the problem for yourself in a later study session.

Both algorithms are designed for use on a computer. They are designed to be efficient for large examples. The examples we give here (to be solved without the use of a computer) are necessarily very simple, and have been chosen to illustrate the working of the algorithms and to demonstrate situations which may occur in practice. Because these examples are simple, it may be possible to solve them more quickly by other methods (for example, by inspection). You will be asked to use the algorithms to solve other examples in the computer activities for *Networks 1*.

#### 3.3 The maximum flow algorithm

This tape sequence describes the maximum flow algorithm for finding a maximum flow and a minimum cut in a basic network. You should listen to this sequence and work through the accompanying frames at the relevant point in your study of *Networks 1*, Section 3.3.

A complete description of the algorithm is given after the tape frames.

Page 10 has been left blank to enable other frames to face each other.

#### 4.1 The feasibility algorithm

In this tape sequence we give an algorithm for determining the feasibility of a given network with lower and upper capacities, and for finding a valid flow if the network proves to be feasible. You should listen to this sequence and work through the accompanying frames at the relevant point in your study of *Networks 1*, Section 4.1.

## BLOCK 2

### Notes for Graphs 2

There is one sequence on the tape for this unit. The heading number 4.2 below refers to the corresponding section in *Graphs 2*.

#### 4.2 Multi-terminal flows

This tape sequence describes the algorithm of Gomory and Hu for obtaining a cut tree, and hence the maximum flow between each pair of vertices, in an *undirected* network.

We first describe an elementary form of the algorithm, and then the full algorithm, which includes the technique of condensing vertices.

Page 32 has been left blank to enable other frames to face each other.

### Notes for Networks 2

There are two sequences on the tape for this unit. The heading number 2.1 below refers to the corresponding section in *Networks 2*.

Each sequence contains a worked problem, followed by a problem for you to do. Solutions to the problems are given in the tape frames.

#### 2.1 Shortest and longest paths

##### The shortest path algorithm

In finding the shortest path from the source  $S$  to the sink  $T$  in a directed network, we assign a temporary label to each vertex, representing the shortest distance from  $S$  to that vertex by all paths so far considered. Eventually each vertex acquires a permanent label, called its *potential*, representing the shortest distance from  $S$  to that vertex. When  $T$  has been assigned a potential, the shortest distance from  $S$  to  $T$  has been found, and we can find the shortest path(s) from  $S$  to  $T$  by tracing back from  $T$  to  $S$  using the potentials.

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##### The longest path algorithm

The algorithm is similar to the shortest path algorithm. However, it applies only to a network which has *no cycles*; otherwise we could go round the cycle an infinite number of times, and the procedure would not terminate.

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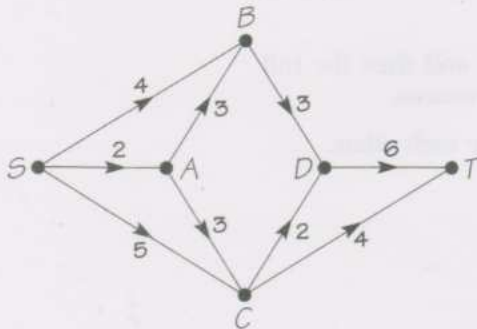


# The maximum flow algorithm

## TRACK 1

### 1 WORKED PROBLEM

Find a maximum flow and a minimum cut in the following basic network



USE

- labelling procedure
- flow-augmenting procedure

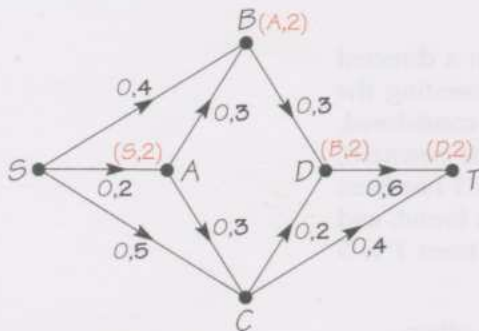
## TRACK 2

### 2 SOLUTION TO WORKED PROBLEM

#### PART A: LABEL VERTICES

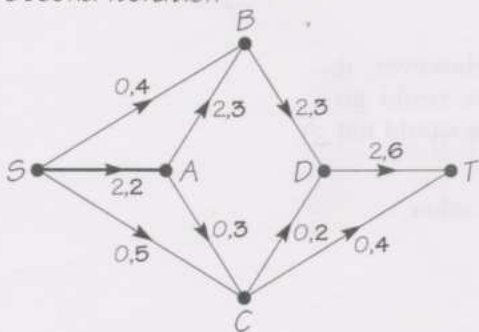
First iteration

Start with zero flow.



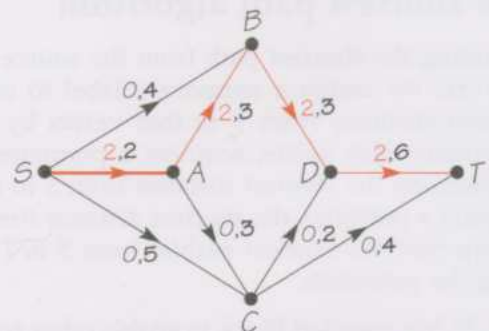
Label A, B, D, T.

Second iteration

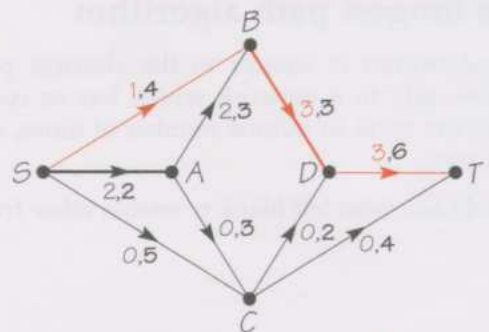


Label B, D, T.

#### PART B: AUGMENT FLOW



Increase flow along SABDT by 2.

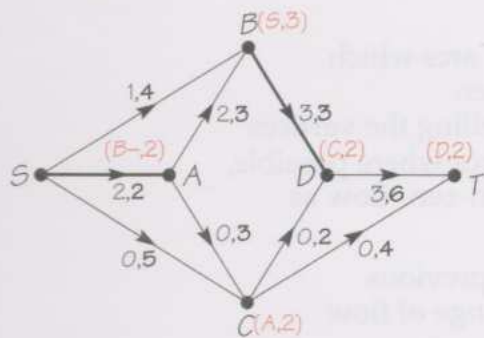


Increase flow along SBDT by 1.

## 2 SOLUTION CONTINUED

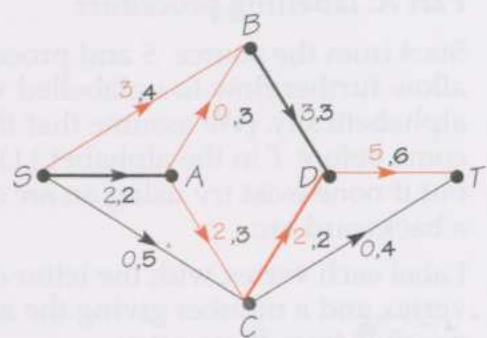
### PART A: LABEL VERTICES

Third iteration



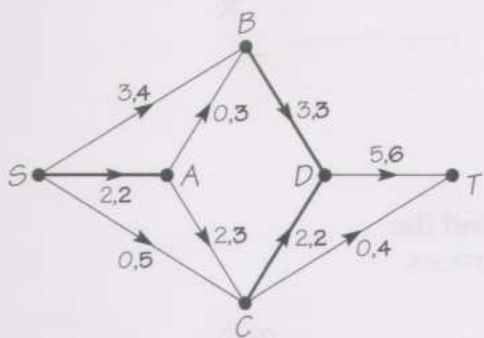
Label B, A, C, D, T.

### PART B: AUGMENT FLOW

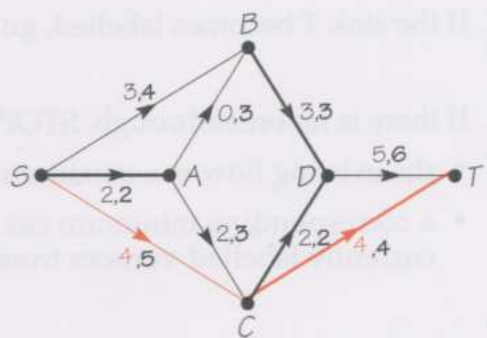


Increase flow along SBACDT by 2.

Fourth iteration

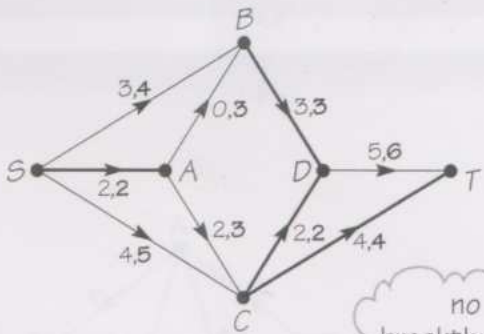


Label B, C, T.



Increase flow along SCT by 4.

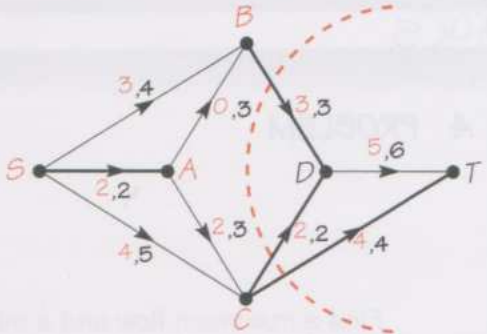
Fifth iteration



Label B, C, A.

No more flow-augmenting paths.

no  
breakthrough



Maximum flow:  $5 + 4 = 9 = 2 + 1 + 2 + 4$ .

Minimum cut  $\{S, A, B, C\}, \{D, T\}$

has capacity  $3 + 2 + 4 = 9$ .

### 3 SUMMARY OF THE ALGORITHM

START with any flow (possibly the zero flow).

#### Part A: labelling procedure

Start from the source  $S$  and proceed along the arcs which allow further flow to unlabelled vertices, chosen alphabetically. (We assume that the letters labelling the vertices come before  $T$  in the alphabet.) Use forward arcs where possible, but if none exist try using an arc carrying a non-zero flow as a backward arc.

Label each vertex with the letter denoting the previous vertex and a number giving the size of the change of flow possible from that vertex.

If a vertex is reached from which no further progress is possible, cross through the current label of that vertex and try backtracking to the vertex previously labelled, and continue from there.

If the sink  $T$  becomes labelled, go to Part B.

breakthrough

If there is no breakthrough, STOP:

- the existing flow is a maximum flow;
- a corresponding minimum cut separates  $S$  and the currently labelled vertices from the other vertices.

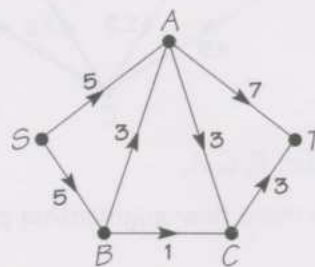
#### Part B: flow-augmenting procedure

Augment the flow along the path of the labelled vertices  $S, \dots, T$  by the largest flow consistent with the labels.

Remove the labels and return to Part A.

### 4 PROBLEM

Find a maximum flow and a minimum cut in the following basic network.

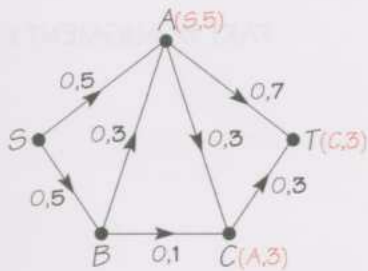




## 5 SOLUTION

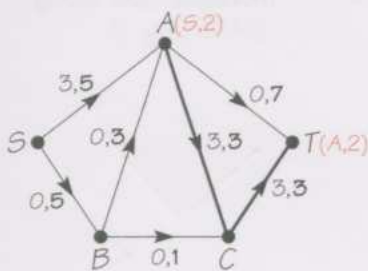
### PART A: LABEL VERTICES

1



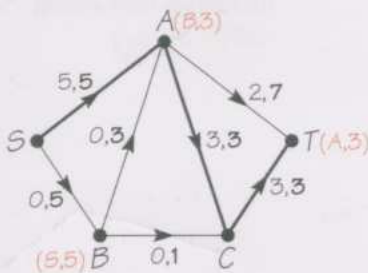
Label A, C, T.

2



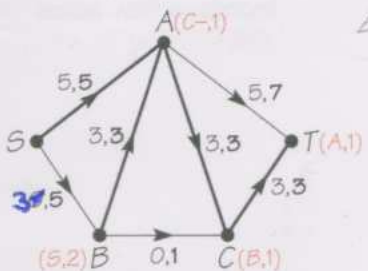
Label A, T.

3



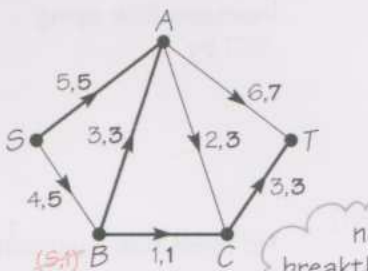
Label B, A, T.

4



Label B, C, A, T.

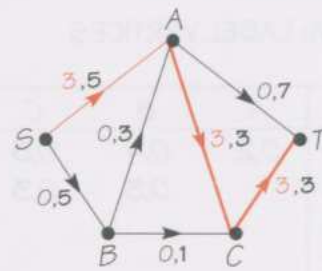
5



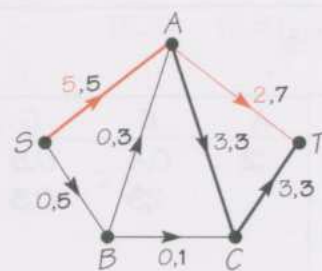
Label B.

No more flow-augmenting paths.

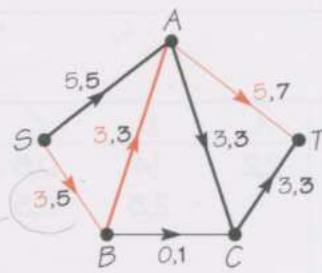
### PART B: AUGMENT FLOW



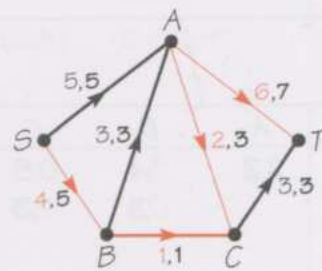
Increase flow along SACT by 3.



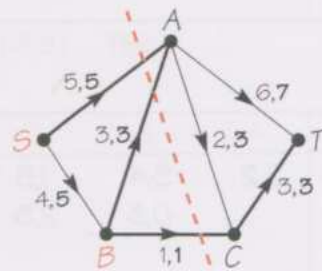
Increase flow along SAT by 2.



Increase flow along SBAT by 3.



Increase flow along SBCAT by 1.



Maximum flow:  $6 + 3 = 9 = 3 + 2 + 3 + 1$ .  
Minimum cut  $\{S, B\}, \{A, C, T\}$   
has capacity  $5 + 3 + 1 = 9$ .

## 6 WORKED PROBLEM

Find a maximum flow and a minimum cut in the basic network in Frame 1.

PART A: LABEL VERTICES

1

	A	B	C	D	T
S	0,2	0,4	0,5		
A		0,3	0,3		
B				0,3	
C				0,2	0,4
D					0,6

(S,2) (A,2) (B,2) (D,2)

2

	A	B	C	D	T
S	2,2	0,4	0,5		
A		2,3	0,3		
B				2,3	
C				0,2	0,4
D					2,6

(S,4) (B,1) (D,1)

3

	A	B	C	D	T
S	2,2	1,4	0,5		
A		2,3	0,3		
B				3,3	
C				0,2	0,4
D					3,6

(S,3) (A,2) (C,2) (D,2)

4

	A	B	C	D	T
S	2,2	3,4	0,5		
A		0,3	2,3		
B				3,3	
C				2,2	0,4
D					5,6

(S,1) (S,5) (C,4)

5

	A	B	C	D	T
S	2,2	3,4	4,5		
A		0,3	2,3		
B				3,3	
C				2,2	4,4
D					5,6

(S,1) (S,1)

PART B: AUGMENT FLOW

Increase flow along **SABDT** by 2.

Increase flow along **SBDT** by 1.

Increase flow along **SBACDT** by 2.

Increase flow along **SCT** by 4.

No more flow-augmenting paths.  
Maximum flow:  $4 + 5 = 9$ .  
Minimum cut  $\{S, A, B, C\}, \{D, T\}$   
has capacity  $3 + 2 + 4 = 9$ .



## 7 PROBLEM

Find a maximum flow and a minimum cut in the basic network in Frame 4.

PART A: LABEL VERTICES

1

	A	B	C	T
S	0,5	0,5		
A			0,3	0,7
B	0,3		0,1	
C				0,3

2

	A	B	C	T
S				
A				
B				
C				

3

	A	B	C	T
S				
A				
B				
C				

4

	A	B	C	T
S				
A				
B				
C				

5

	A	B	C	T
S				
A				
B				
C				

PART B: AUGMENT FLOW

Increase flow along  
..... by .....

Increase flow along  
..... by .....

Increase flow along  
..... by .....

Increase flow along  
..... by .....

No more flow-augmenting paths.  
Maximum flow: .....  
Minimum cut .....  
has capacity .....

# 7. PROBLEM

Find a minimum flow and a maximum flow in the flow network in Figure 8.

PART A: LABEL VERTICES

	A	B	C	T
1	0	0	0	0
2	0	0	0	0
3	0	0	0	0
4	0	0	0	0

	A	B	C	T
1				
2				
3				
4				

	A	B	C	T
1				
2				
3				
4				

	A	B	C	T
1				
2				
3				
4				

	A	B	C	T
1				
2				
3				
4				

PART B: MINIMUM FLOW

Minimum flow: \_\_\_\_\_

Minimum flow: \_\_\_\_\_

Minimum flow: \_\_\_\_\_

Minimum flow: \_\_\_\_\_

For each flow augmenting path:  
 Maximum flow: \_\_\_\_\_  
 Minimum flow: \_\_\_\_\_  
 Flow capacity: \_\_\_\_\_

# 8 SOLUTION TO PROBLEM USING TABULAR METHOD

## PART A: LABEL VERTICES

## PART B: AUGMENT FLOW

1

	A	B	C	T
S	0,5	0,5		
A			0,3	0,7
B	0,3		0,1	
C				0,3

(S,5) (A,3) (C,3)

Increase flow along  
SACT by 3.

2

	A	B	C	T
S	3,5	0,5		
A			3,3	0,7
B	0,3		0,1	
C				3,3

(S,2) (A,2)

Increase flow along  
SAT by 2.

3

	A	B	C	T
S	5,5	0,5		
A			3,3	2,7
B	0,3		0,1	
C				3,3

(B,3) (S,5) (A,3)

Increase flow along  
SBAT by 3.

4

	A	B	C	T
S	5,5	3,5		
A			3,3	5,7
B	3,3		0,1	
C				3,3

(S,2) (B,1) (A,1) (C,-1)

Increase flow along  
SBCAT by 1.

5

	A	B	C	T
S	5,5	4,5		
A			2,3	6,7
B	3,3		1,1	
C				3,3

(S,1)

No more flow-augmenting paths.  
Maximum flow:  $6 + 3 = 9$ .  
Minimum cut  $\{S, B\}, \{A, C, T\}$   
has capacity  $5 + 3 + 1 = 9$ .



# Maximum flow algorithm

## Part A: labelling procedure

START with a flow found by inspection, and with all the vertices unlabelled.

STEP 1 Choose an unsaturated arc  $SV_1$  and assign to the vertex  $V_1$  the label  $(S, a_1)$ , where  $a_1$  is the amount of flow needed to saturate the arc  $SV_1$ .

STEP 2 Choose an unlabelled vertex  $V_2$  such that  
*either* (a)  $V_1V_2$  is an unsaturated arc  
*or* (b)  $V_2V_1$  is an arc with a non-zero flow.

In case (a), assign to the vertex  $V_2$  the label  $(V_1, a)$ , where  $a$  is either  $a_2$ , the amount needed to saturate  $V_1V_2$ , or  $a_1$ , whichever is the smaller.

In case (b), assign to the vertex  $V_2$  the label  $(V_1^-, a)$ , where  $a$  is either  $a_2$ , the amount of flow in  $V_2V_1$ , or  $a_1$ , whichever is the smaller.

Continue in this way, choosing vertices  $V_3, V_4, \dots$

At each stage, choose the vertex  $V_{k+1}$  to be any unlabelled vertex such that

*either* (a)  $V_kV_{k+1}$  is an unsaturated arc  
*or* (b)  $V_{k+1}V_k$  is an arc with a non-zero flow.

In case (a), assign to the vertex  $V_{k+1}$  the label  $(V_k, a)$ , where  $a$  is either  $a_{k+1}$ , the amount needed to saturate  $V_kV_{k+1}$ , or  $a_k$ , whichever is the smaller.

In case (b), assign to the vertex  $V_{k+1}$  the label  $(V_k^-, a)$ , where  $a$  is either  $a_{k+1}$ , the amount of flow in  $V_{k+1}V_k$ , or  $a_k$ , whichever is the smaller.

STEP 3 Continue in this way until

*either* (1) the vertex  $T$  becomes labelled (breakthrough)  
*or* (2) there is no vertex  $V_{k+1}$  with property (a) or (b).

In case (1), a flow-augmenting path has been found, so proceed to Part B.

In case (2), backtrack along the path, crossing through the labels of vertices which do not lie on a flow-augmenting path, until vertex  $V_i$  is reached with the property that either there is an unsaturated arc  $V_iW$  or there is an arc  $WV_i$  carrying a non-zero flow, where  $W$  is as yet unlabelled. If such a vertex  $V_i$  is found, regard  $W$  as  $V_{i+1}$  and label it as in Step 2.

STEP 4 Continue in this way, labelling vertices where possible and backtracking where necessary until

*either* (a) vertex  $T$  becomes labelled (breakthrough)  
*or* (b) no further labelling is possible.

In case (a), proceed directly to Part B.

In case (b), no increase in flow is possible, and a maximum flow has already been obtained. The arcs separating the source  $S$  and the labelled vertices from the other vertices form a minimum cut.  
STOP.

In the audio-tape and computer activities, we proceed systematically — considering vertices in alphabetical order, and choosing forward arcs rather than backward arcs where possible. However, these are not essential features of the algorithm. When working on small networks without the use of a computer, it is usually quicker to use the general version of the algorithm given here.

## Part B: flow-augmenting procedure

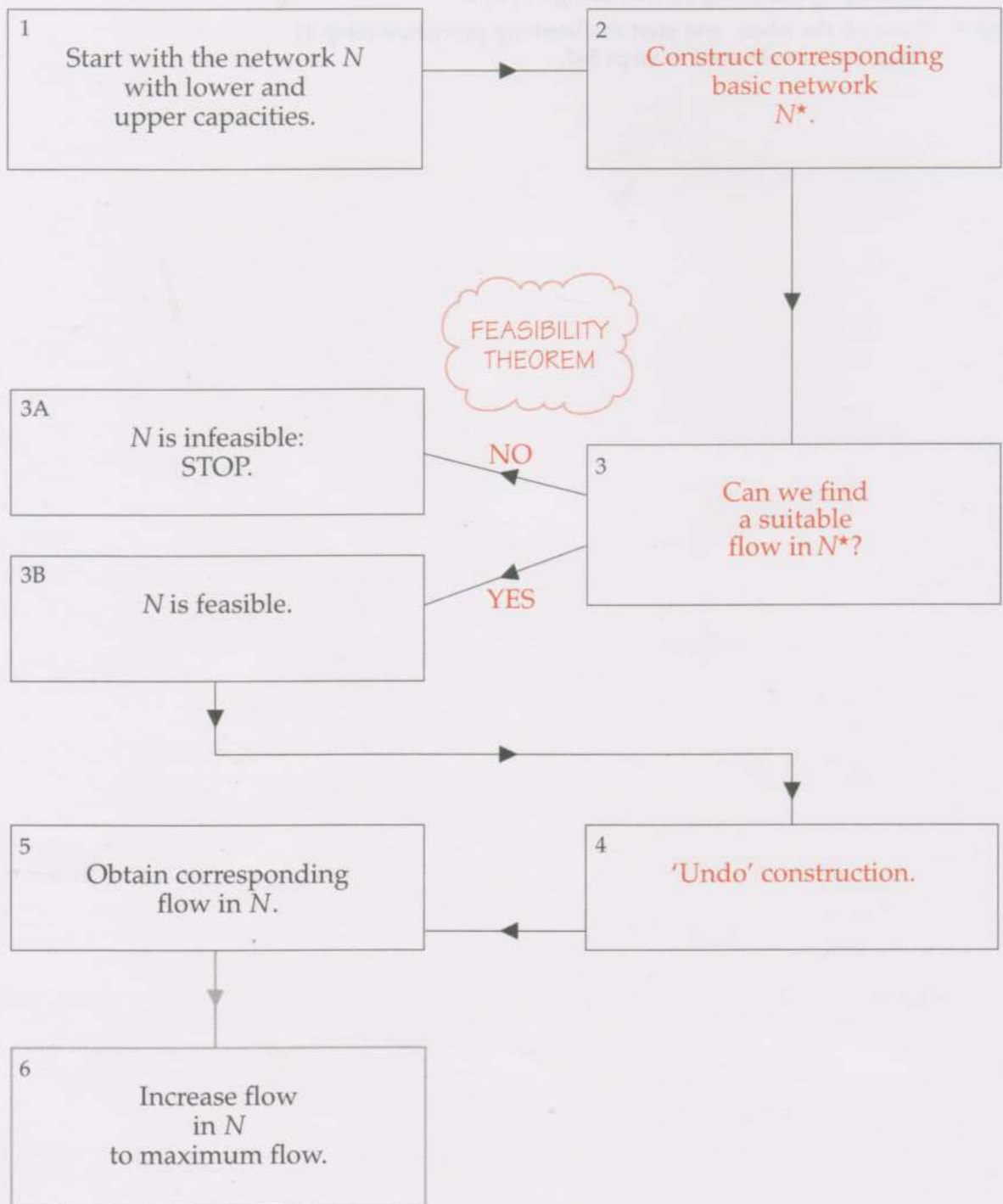
The following procedure is used when the sink  $T$  has been labelled.

- STEP 5 Look at the label of  $T$ . It will be of the form  $(V_k, a)$ , for some vertex  $V_k$  and some positive number  $a$ . Increase the flow along  $V_k T$  by  $a$ .
- STEP 6 Look at the label of  $V_k$ . It will be of the form  $(V_{k-1}, a)$  or  $(V_{k-1}, -a)$  for some vertex  $V_{k-1}$ . In the former case, increase the flow along  $V_{k-1} V_k$  by  $a$ , and in the latter case decrease the flow along  $V_k V_{k-1}$  by  $a$ .
- STEP 7 Repeat Step 7 successively for the vertices  $V_{k-1}, V_{k-2}, \dots, V_1$ , and conclude by increasing the flow along  $S V_1$  by  $a$ .
- STEP 8 Erase all the labels, and start the labelling procedure (Step 1) using the flow obtained in Steps 5–7.

## The feasibility algorithm

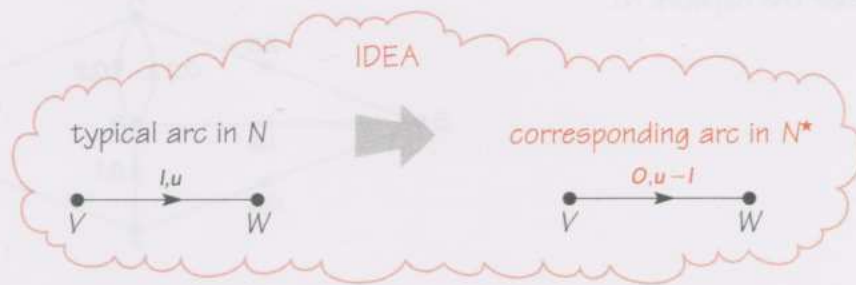
TRACK 9

### 1 OUTLINE OF THE ALGORITHM

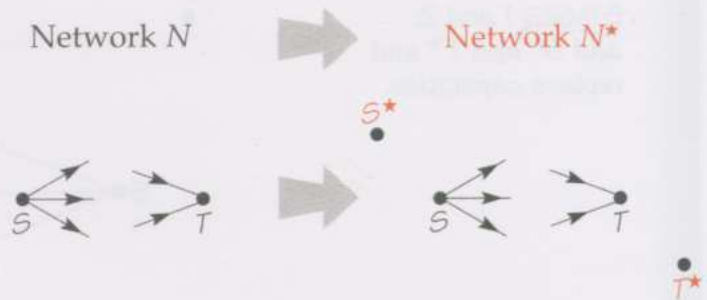




## 2 CONSTRUCTION OF $N^*$



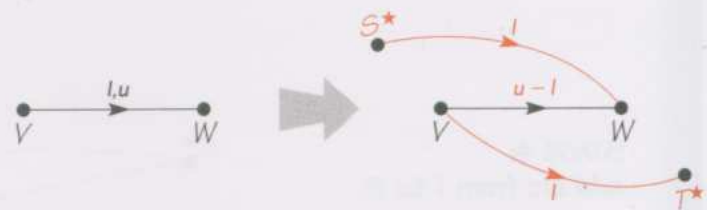
STAGE 1: take vertices of  $N^*$  to be vertices of  $N$  together with a new source  $S^*$  and a new sink  $T^*$ .



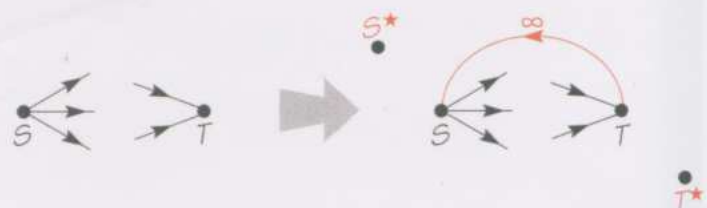
STAGE 2: for each arc  $VW$ , replace the capacities  $l, u$  by  $u-l$ .



STAGE 3: for each arc  $VW$  with lower capacity  $l$ , add two new arcs: an arc  $S^*W$  and an arc  $VT^*$ , each with capacity  $l$ .

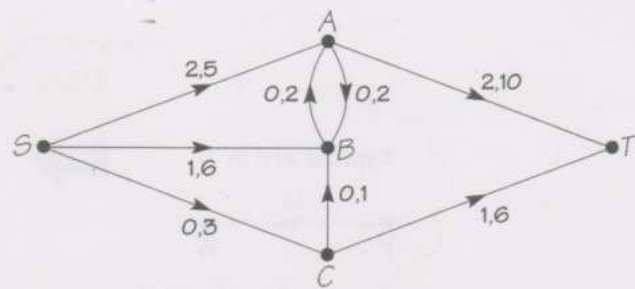


STAGE 4: add a new arc from  $T$  to  $S$  with infinite capacity.

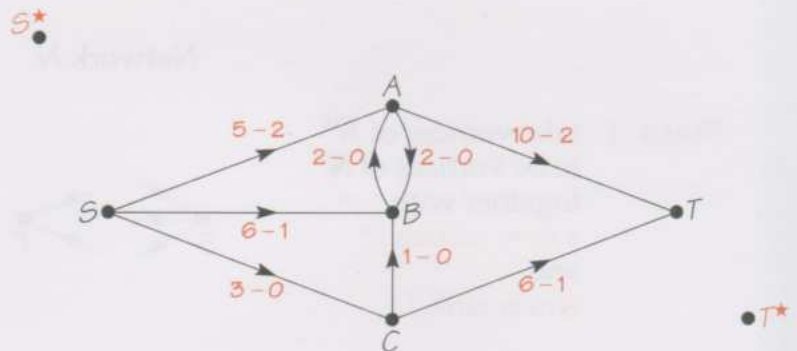


### 3 EXAMPLE

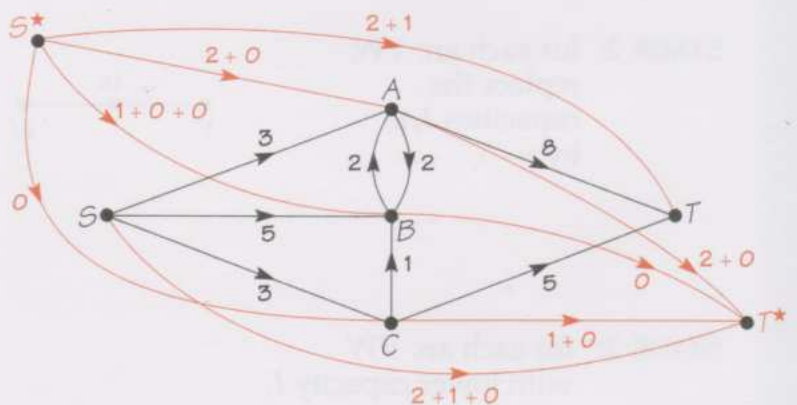
Consider the network  $N$ :



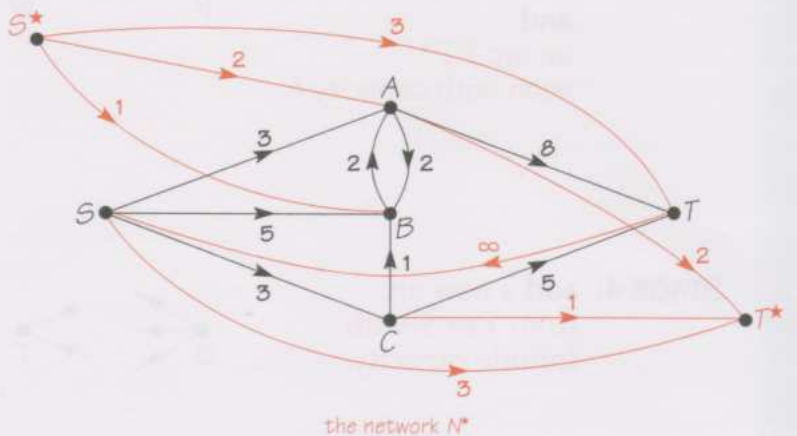
STAGES 1 and 2:  
add  $S^*$  and  $T^*$  and  
replace capacities.



STAGE 3:  
for each arc  $VW$ ,  
add arcs  $S^*W$  and  $VT^*$ .



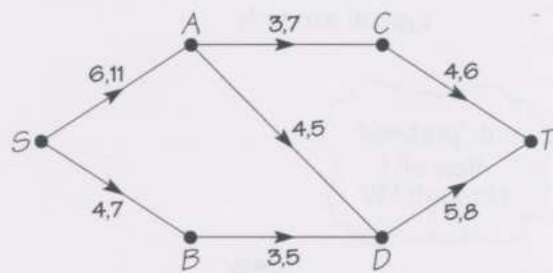
STAGE 4:  
add arc from  $T$  to  $S$ .



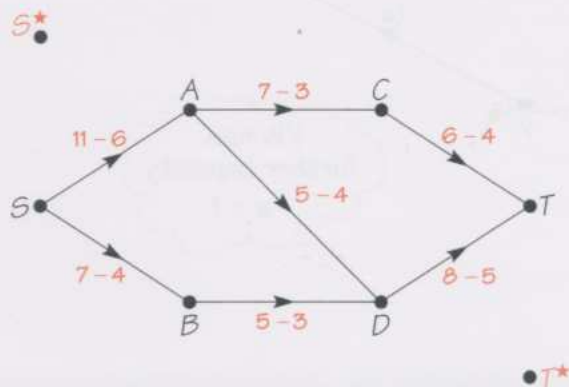
the network  $N^*$

#### 4 PROBLEM

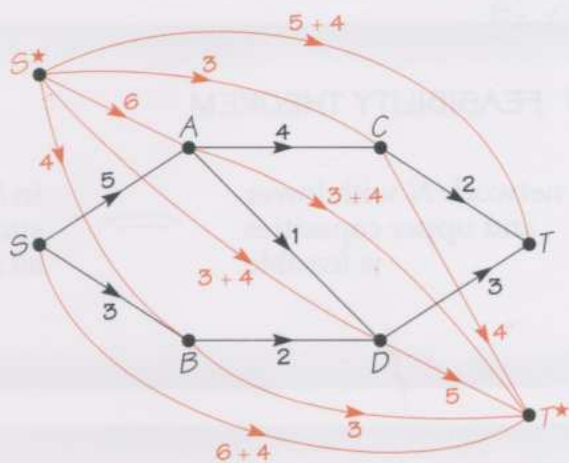
Consider the network  $N$ :



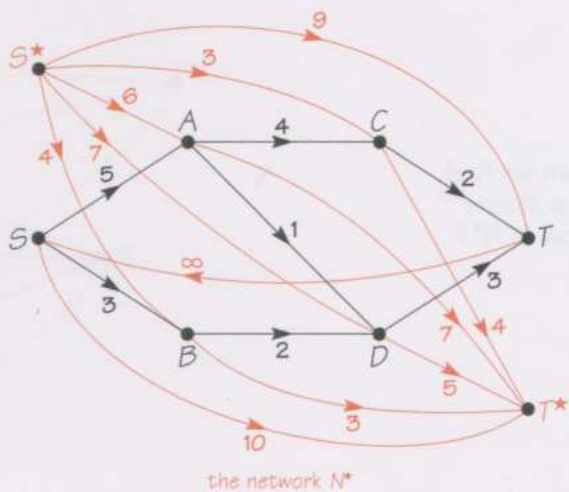
STAGES 1 and 2:



STAGE 3:

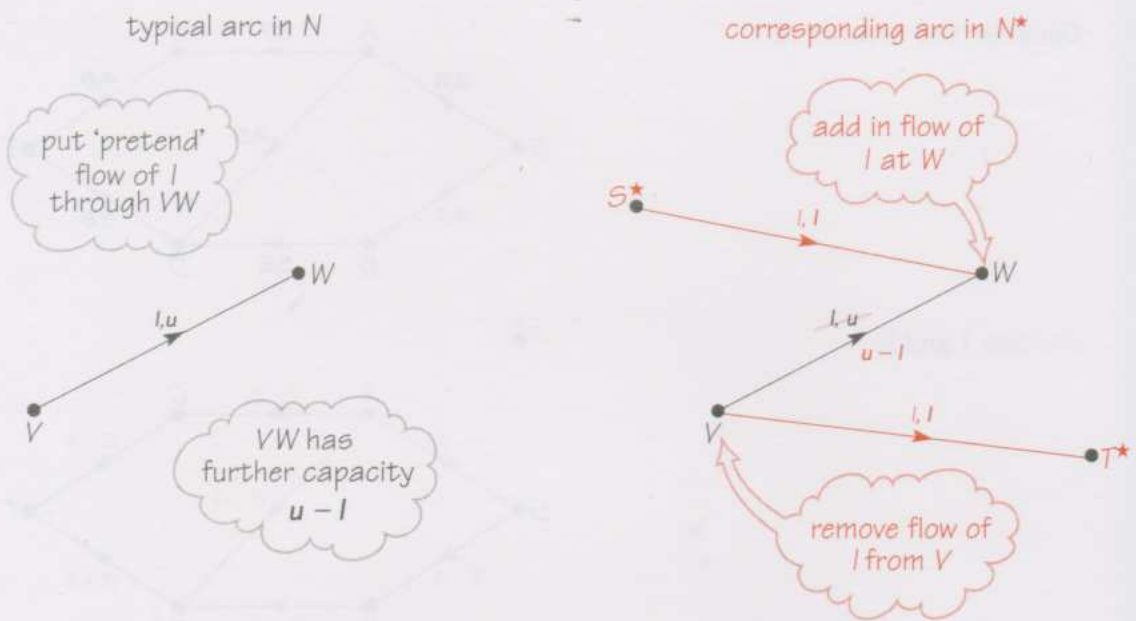


STAGE 4:





## 5 UNDERLYING IDEA



## TRACK 14

## 6 FEASIBILITY THEOREM

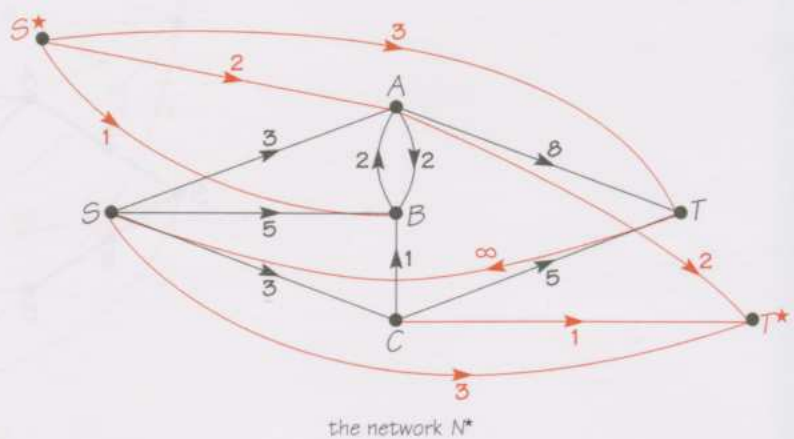
a network  $N$  with lower and upper capacities is feasible



in  $N^*$  there is a flow from  $S^*$  to  $T^*$  such that all arcs out of  $S^*$  and all arcs into  $T^*$  are saturated

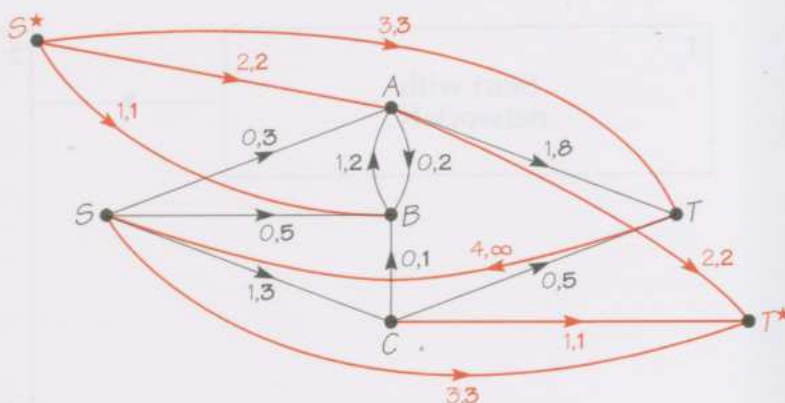
## 7 EXAMPLE CONTINUED

Can we find a suitable flow in  $N^*$ ?



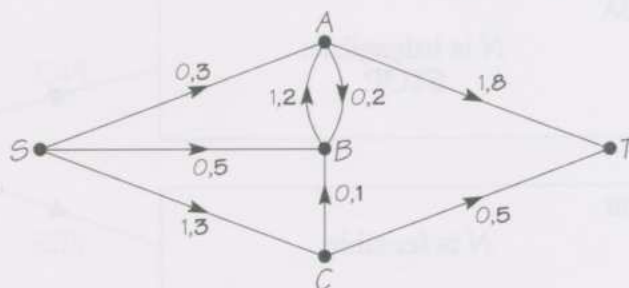
# 8 EXAMPLE CONTINUED

We can find a flow in  $N^*$  such that all arcs out of  $S^*$  and all arcs into  $T^*$  are saturated:  
 $N$  is feasible.

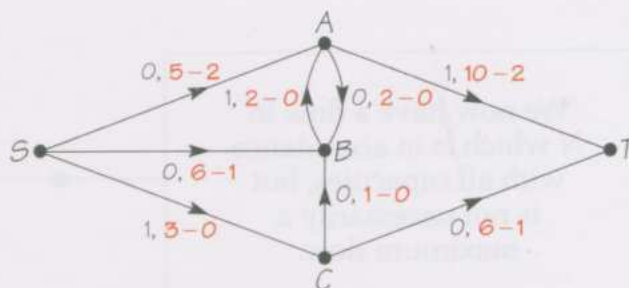


a suitable flow of value 6 in  $N^*$

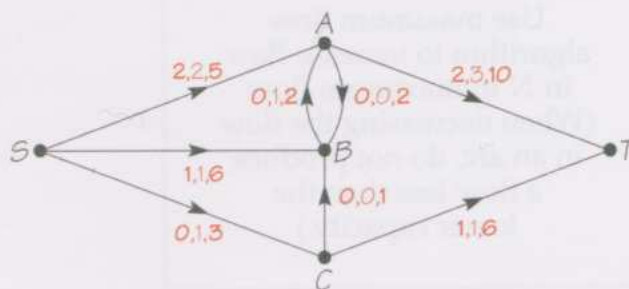
'Undo' stages 1, 3 and 4:  
 remove  $S^*$  and  $T^*$ , and  
 all extra arcs.



Next, we 'undo' stage 2.

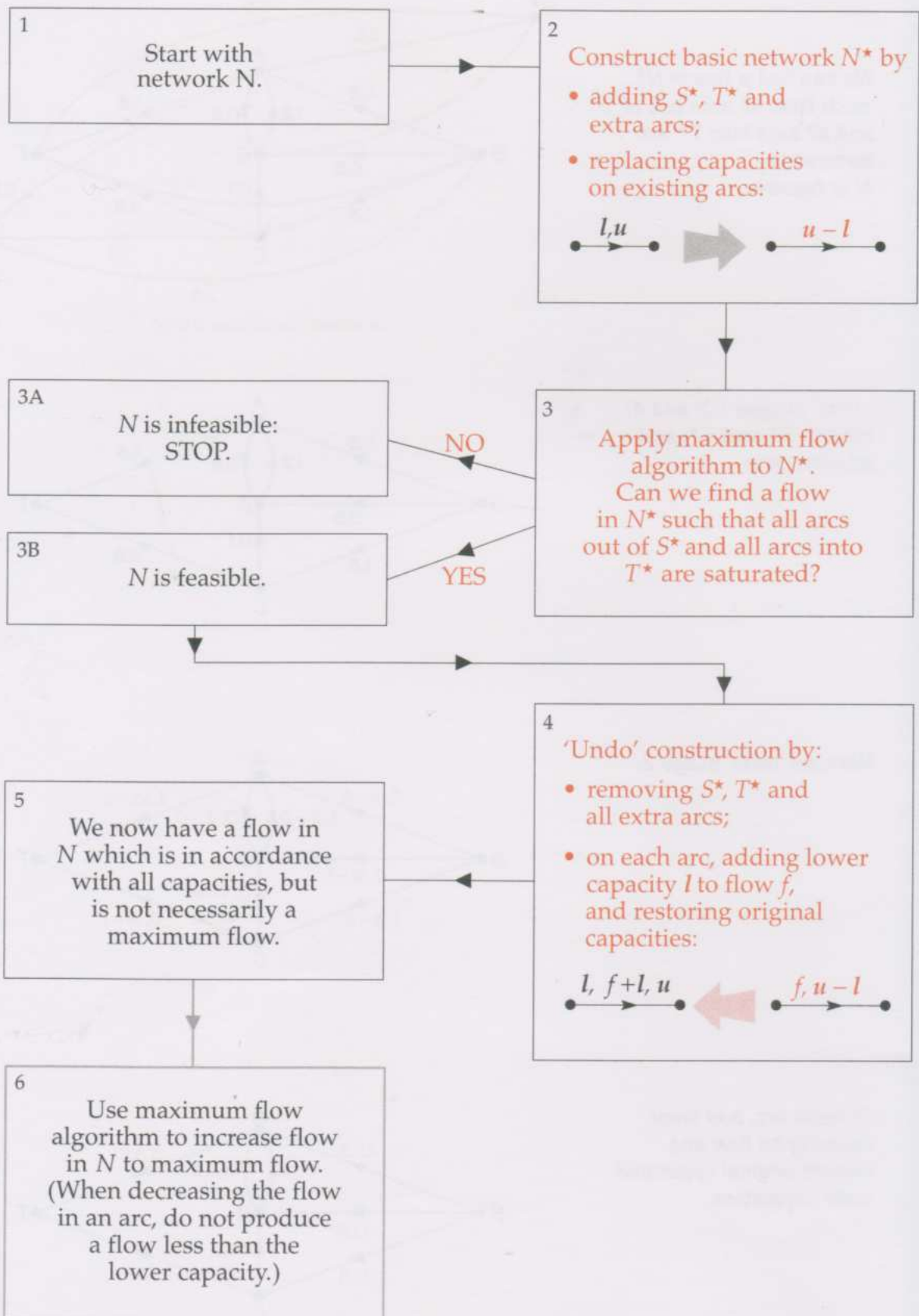


On each arc, add lower  
 capacity to flow and  
 restore original upper and  
 lower capacities.



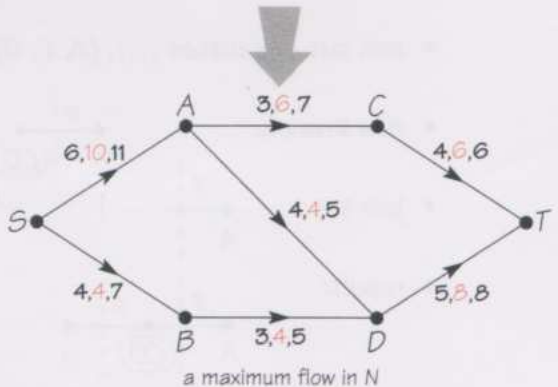
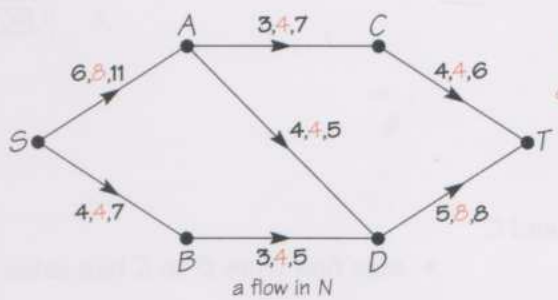
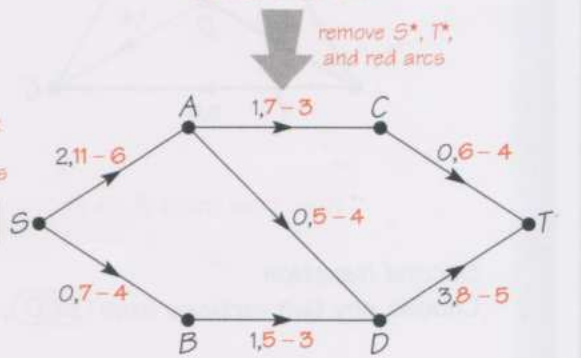
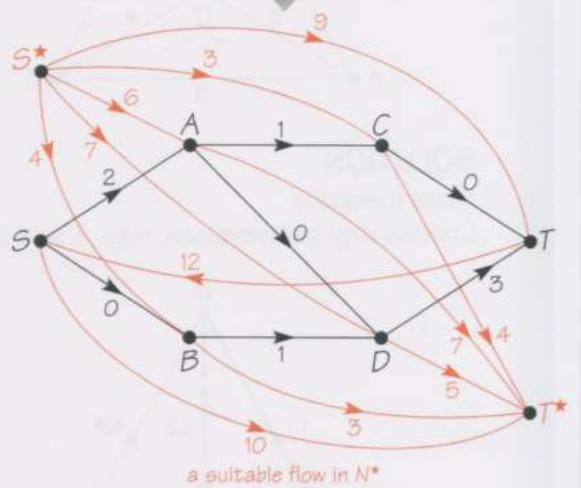
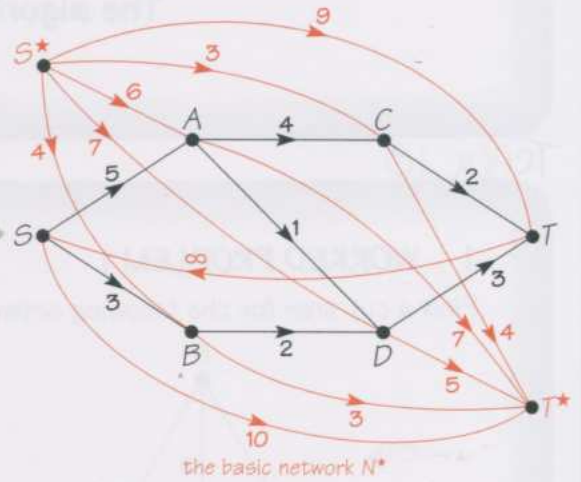
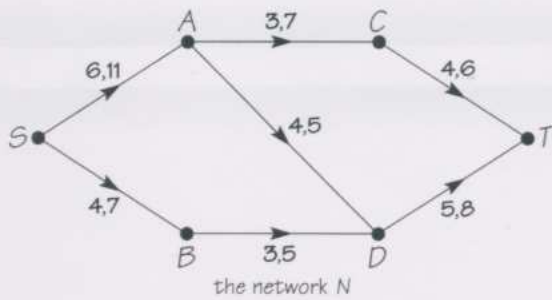
a flow of value 4 in  $N$

## 9 THE ALGORITHM





10 PROBLEM CONTINUED

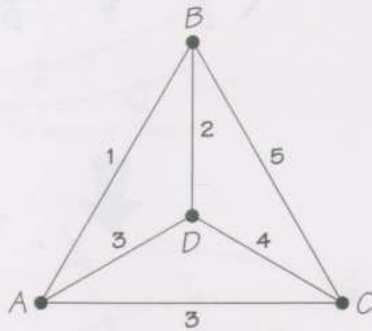


# The algorithm of Gomory and Hu

TRACK 18

## 1 WORKED PROBLEM 1

Find a cut tree for the following network.

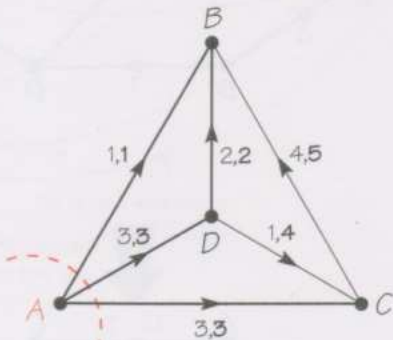


4 vertices,  
so 3 iterations

### SOLUTION

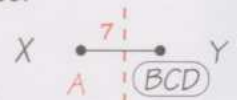
First iteration

Choose any two vertices, say A and B.



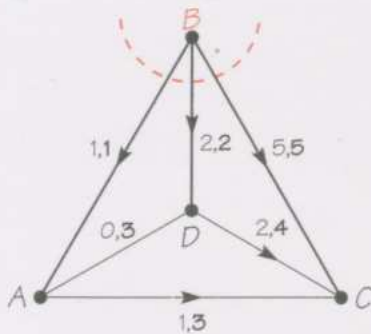
max flow from A to B

- max flow from A to B has value 7;
- min cut separates  $\{A\}$ ,  $\{B, C, D\}$ ;
- first branch of tree:



Second iteration

Choose any two vertices from  $\{B, C, D\}$ , say B and C.



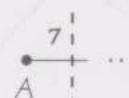
max flow from B to C

- max flow from B to C has value 8;
- min cut separates  $\{B\}$ ,  $\{A, C, D\}$ ;

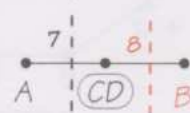
new branch:



join to:



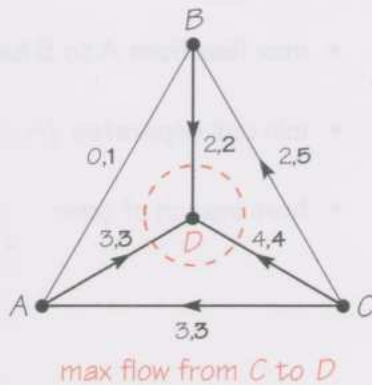
result:



## 1 SOLUTION CONTINUED

Third iteration

Consider the two vertices  $C$  and  $D$ .



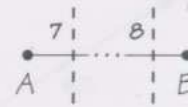
- max flow from  $C$  to  $D$  has value 9;

- min cut separates  $\{D\}$ ,  $\{A, B, C\}$ ;

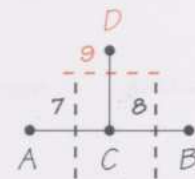
- new branch:



- join to:



- cut tree:



TRACK 19

## 2 OUTLINE OF THE ALGORITHM

General iteration

Choose two vertices.

To find the branch in the cut tree joining the two chosen vertices:

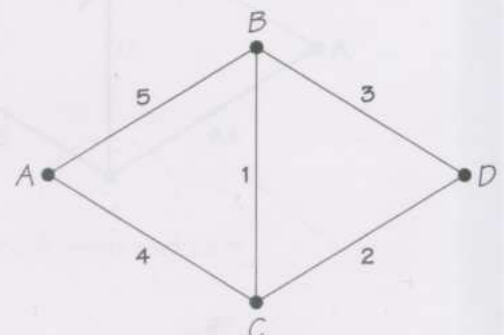
- find a maximum flow from the first vertex to the second, and find a corresponding minimum cut in the network;
- note how the minimum cuts splits all the vertices into two disjoint subsets;
- draw the corresponding new branch of the cut tree;
- draw the existing part(s) of the cut tree that the new branch must be joined to;
- join the new branch to the existing part of the cut tree.

choosing vertices

first iteration: choose any two;  
later: choose two from any  
generalized vertex

## 3 PROBLEM 1

Use the algorithm to find a cut tree for the following network.  
First find a maximum flow from  $A$  to  $B$ .

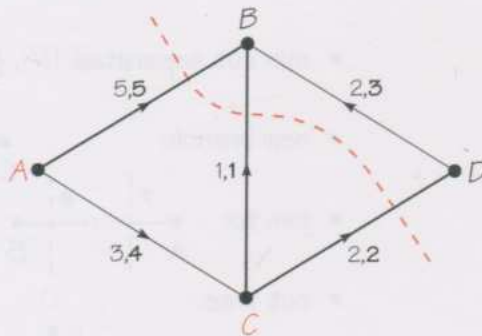




#### 4 SOLUTION TO PROBLEM 1

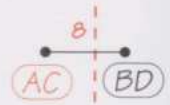
First iteration

Choose vertices A and B.



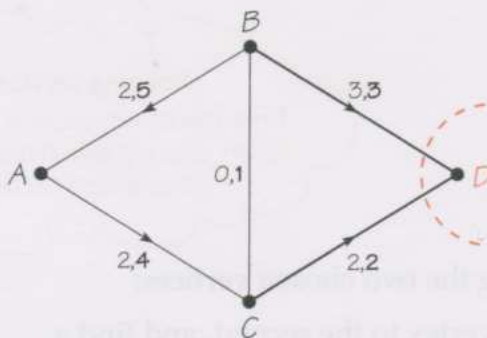
max flow from A to B

- max flow from A to B has value 8;
- min cut separates  $\{A, C\}$ ,  $\{B, D\}$ ;
- first branch of tree:



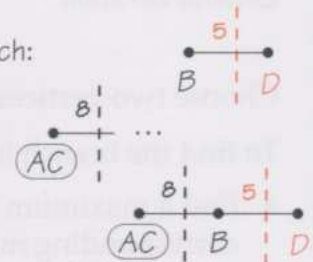
Second iteration

Split up generalized vertex (BD).



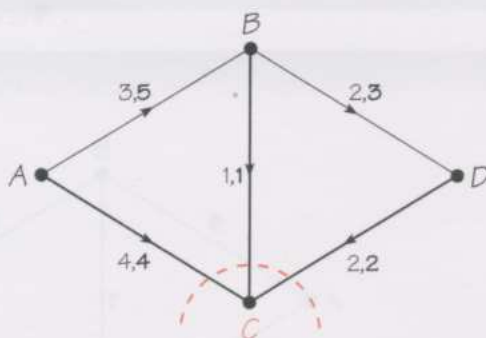
max flow from B to D

- max flow from B to D has value 5;
- min cut separates  $\{D\}$ ,  $\{A, B, C\}$ ;
- new branch:
- join to:
- result:



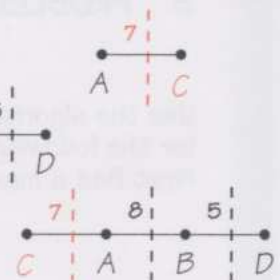
Third iteration

Split up generalized vertex (AC).



max flow from A to C

- max flow from A to C has value 7;
- min cut separates  $\{C\}$ ,  $\{A, B, D\}$ ;
- new branch:
- join to: ...
- cut tree:

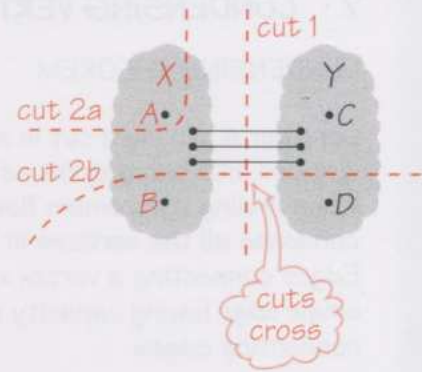




## 5 DOES IT ALWAYS WORK?

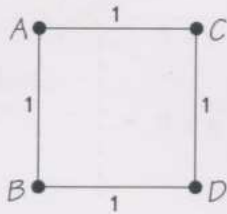
Suppose the first minimum cut (cut 1) separates the vertices into two disjoint sets  $X$  and  $Y$ .

Must the second minimum cut, separating vertices  $A$  and  $B$  in  $X$ , necessarily keep  $Y$  entirely on one side of the cut, so that  $Y$  may be joined to either  $A$  or  $B$ , as appropriate?

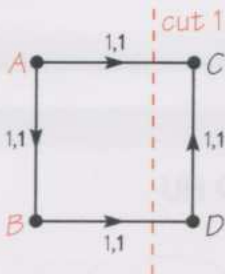


## TRACK 22

### 6 EXAMPLE



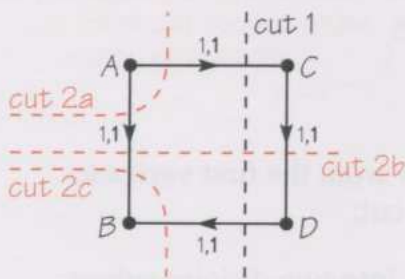
First iteration  
Choose vertices  $A$  and  $C$ .



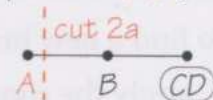
- max flow from  $A$  to  $C$  has value 2;
- min cut 1 separates  $\{A, B\}, \{C, D\}$ ;
- first branch of tree:



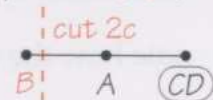
Second iteration  
Split up  $(AB)$ .



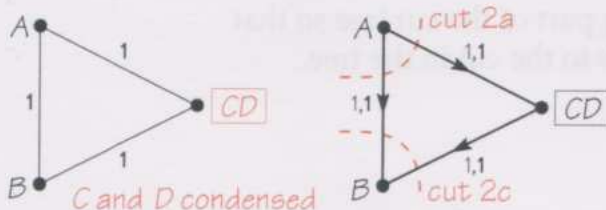
- max flow from  $A$  to  $B$  has value 2;
- min cut 2a separates  $\{A\}, \{B, C, D\}$ :



- min cut 2b separates  $\{A, C\}, \{B, D\}$ ; ✗
- min cut 2c separates  $\{B\}, \{A, C, D\}$ :



We can avoid minimum cuts crossing by **condensing vertices**:



- max flow from  $A$  to  $B$  still has value 2;
- min cut 2b is no longer possible;
- min cuts 2a and 2c remain.

## 7 CONDENSING VERTICES

### CONDENSING THEOREM

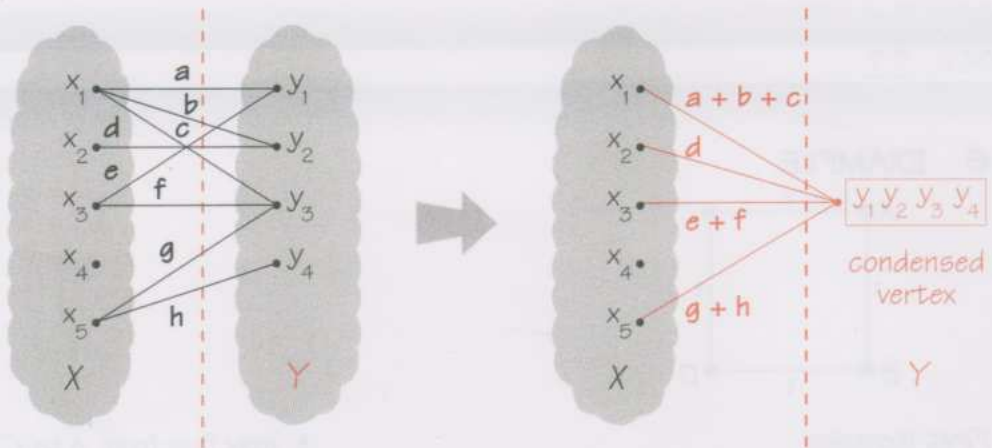
Consider a minimum cut in an undirected network  $N$  separating the vertices into two disjoint sets  $X$  and  $Y$ .

When finding a maximum flow between any two vertices in  $X$ , we may condense all the vertices in  $Y$  to a single vertex (and vice versa).

Edges connecting a vertex  $x$  in  $X$  to any vertex in  $Y$  are replaced by a single edge having capacity equal to the sum of the capacities of the connecting edges.

avoids  
cuts  
crossing

simplifies  
calculations



## 8 OUTLINE OF FULL ALGORITHM OF GOMORY AND HU

### General iteration

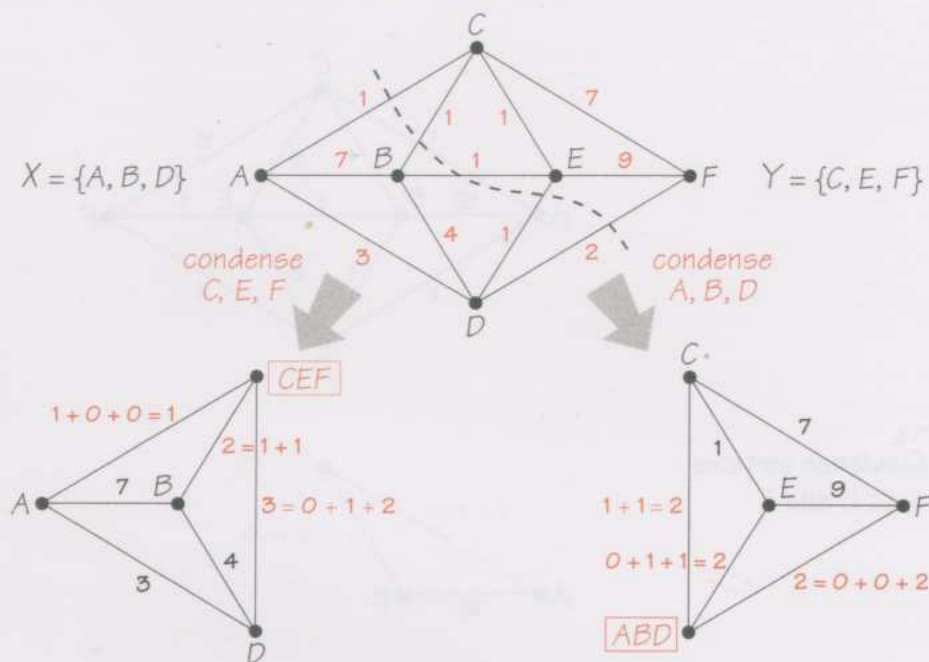
Choose two vertices from a generalized vertex.

To find a new branch of the cut tree:

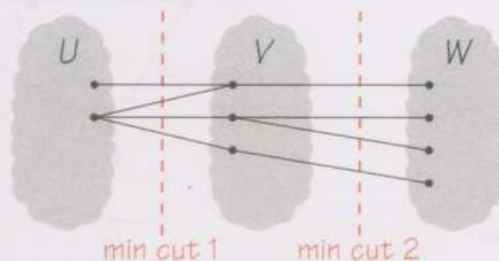
- apply the condensing theorem appropriately;
- in the condensed network, find a maximum flow from the first vertex to the second, and find a corresponding minimum cut;
- note how the minimum cut splits all the vertices into two disjoint subsets;
- draw the corresponding new branch of the cut tree;
- join the new branch to the existing part of the cut tree so that the cut in the network corresponds to the cut in the tree.

choosing vertices  
first iteration: choose any two;  
later: choose two from any  
generalized vertex

## 9 EXAMPLE OF CONDENSING VERTICES



## 10 USING THE CONDENSING THEOREM TWICE



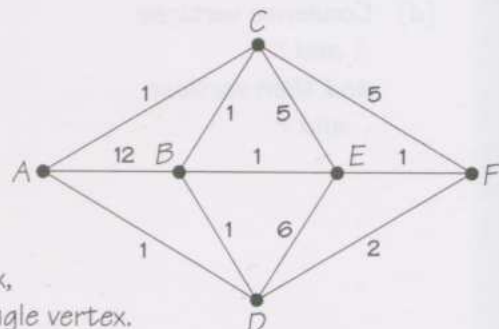
When finding a maximum flow between 2 vertices in  $V$ , we may:

- condense all the vertices in  $U$  (applying the theorem to min cut 1);
- condense all the vertices in  $W$  (applying the theorem to min cut 2).

## 11 PROBLEM 2

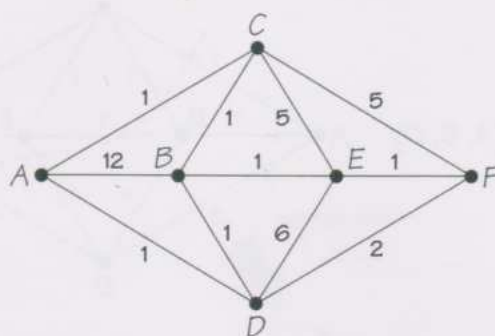
In the given network,

- condense the vertices  $C, D, E$  and  $F$ ;
- condense the vertices  $A$  and  $B$ ;
- condense the vertices  $A, B, D$  and  $E$ ;
- condense the vertices  $A$  and  $B$  to a single vertex, and then condense the vertices  $C$  and  $F$  to a single vertex.

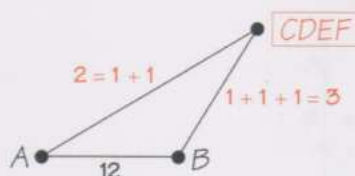




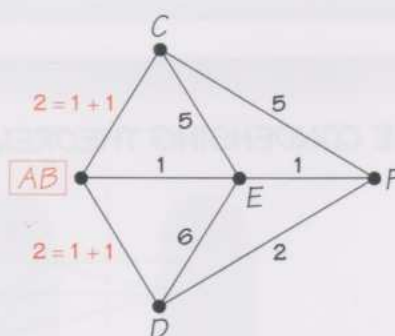
## 12 SOLUTION TO PROBLEM 2



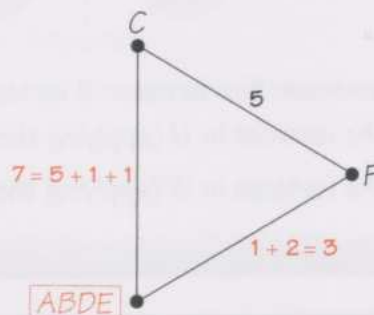
- (a) Condense vertices  
C, D, E and F.



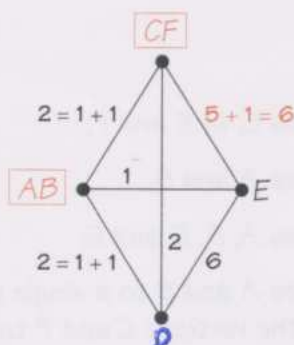
- (b) Condense vertices  
A and B.



- (c) Condense vertices  
A, B, D and E.



- (d) Condense vertices  
A and B,  
and then vertices  
C and F.

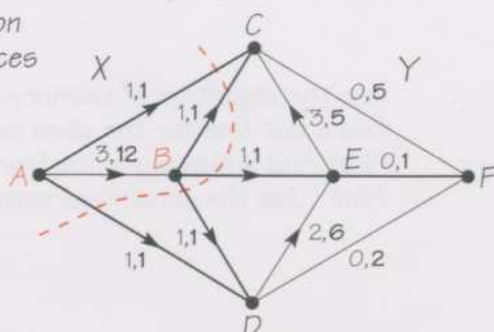




### 13 WORKED PROBLEM 2

Find a cut tree for the network in Frame 11.

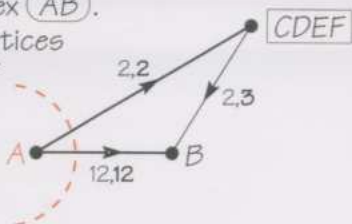
First iteration  
Choose vertices  
A and C.



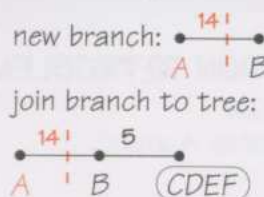
- max flow from A to C has value 5;
- min cut separates  $\{A, B\}$ ,  $\{C, D, E, F\}$ ;
- first branch of tree:



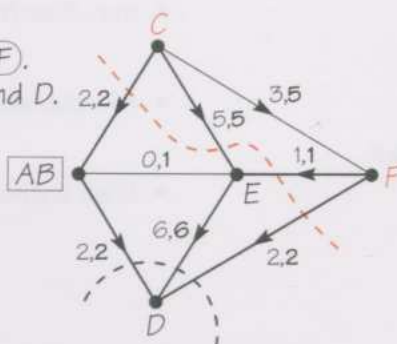
Second iteration  
Split up vertex (AB).  
Condense vertices  
C, D, E and F



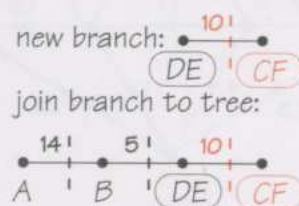
- max flow from A to B has value 14;
- min cut separates  $\{A\}$ ,  $\{B, C, D, E, F\}$ ;
- new branch:
- join branch to tree:



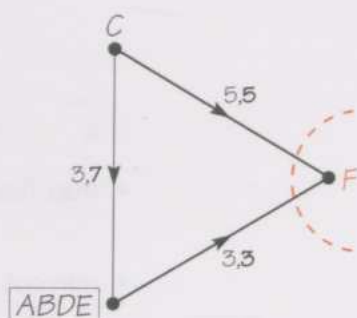
Third iteration  
Split up vertex (CDEF).  
Choose vertices C and D.  
Condense vertices  
A and B.



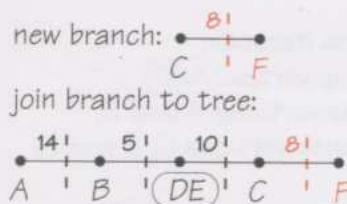
- max flow from C to D has value 10;
- min cut separates  $\{C, F\}$ ,  $\{A, B, D, E\}$ ;
- new branch:
- join branch to tree:



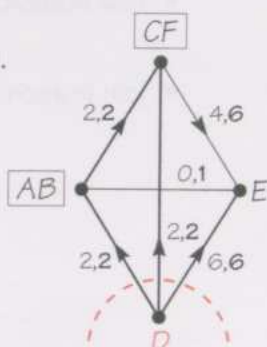
Fourth iteration  
Split up vertex (CF).  
Condense vertices  
A, B, D and E.



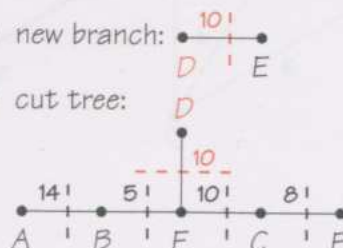
- max flow from C to F has value 8;
- min cut separates  $\{F\}$ ,  $\{A, B, C, D, E\}$ ;
- new branch:
- join branch to tree:



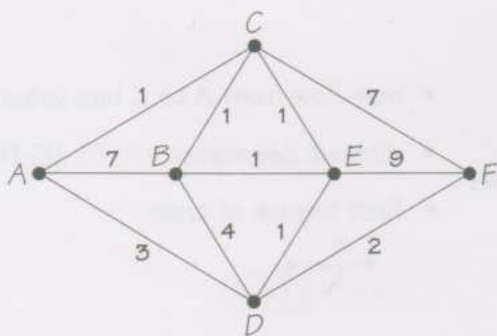
Fifth iteration  
Split up vertex (DE).  
Condense vertices  
A and B,  
and then vertices  
C and F.



- max flow from D to E has value 10;
- min cut separates  $\{D\}$ ,  $\{A, B, C, E, F\}$ ;
- new branch:
- cut tree:



### 14 PROBLEM 3

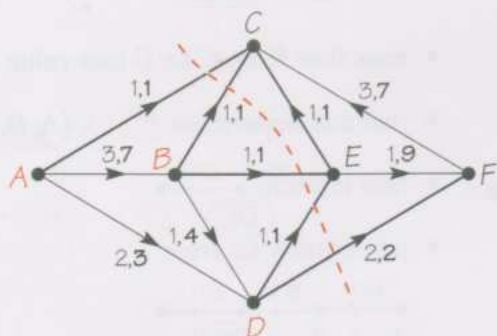


Use the algorithm of Gomory and Hu to find a cut tree for the given network. First find a maximum flow from A to C. Hint Use the condensed vertices in Frame 9.

### TRACK 30

### 15 SOLUTION TO PROBLEM 3

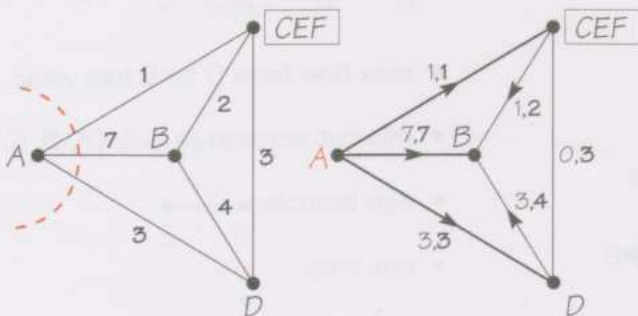
First iteration  
Choose vertices A and C.



- max flow from A to C has value 6;
- min cut separates  $\{A, B, D\}, \{C, E, F\}$ ;
- first branch of tree:



Second iteration  
Split up vertex ABD.  
Choose vertices A and B.  
Condense vertices C, E and F.



- max flow from A to B has value 11;
- min cut separates  $\{A\}, \{B, C, D, E, F\}$ ;
- new branch: A BD
- join branch to tree:

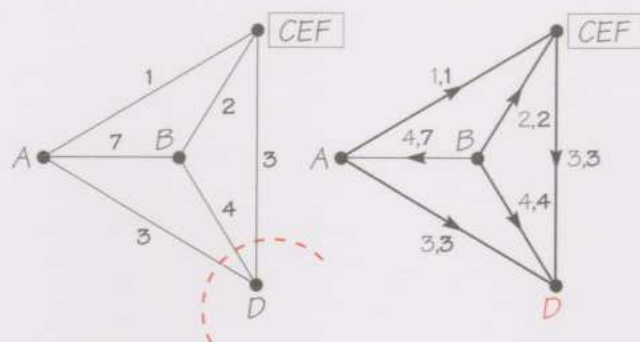


## 15 SOLUTION CONTINUED

Third iteration

Split up vertex  $(BD)$ .

Condense vertices  $C, E$  and  $F$ .

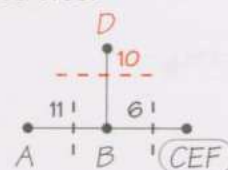


- max flow from  $B$  to  $D$  has value  $10$ ;

- min cut separates  $\{D\}, \{A, B, C, E, F\}$ ;

- new branch:  $\overset{10}{B} \dashv \overset{1}{D}$

- join branch to tree:

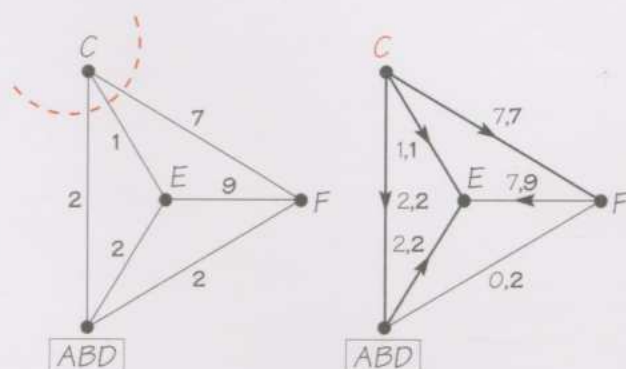


Fourth iteration

Split up vertex  $(CEF)$ .

Choose vertices  $C$  and  $E$ .

Condense vertices  $A, B$  and  $D$ .

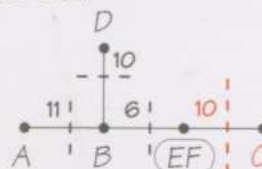


- max flow from  $C$  to  $E$  has value  $10$ ;

- min cut separates  $\{C\}, \{A, B, D, E, F\}$ ;

- new branch:  $\overset{10}{C} \dashv \overset{1}{EF}$

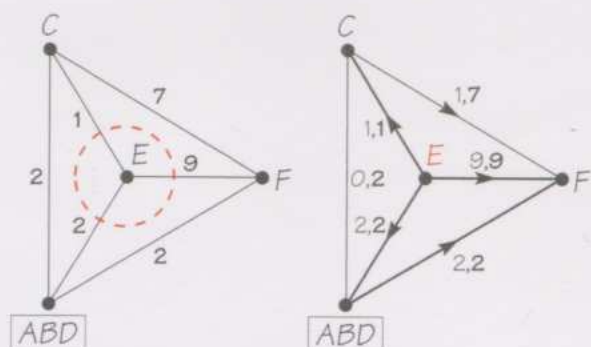
- join branch to tree:



Fifth iteration

Split up vertex  $(EF)$ .

Condense vertices  $A, B$  and  $D$ .

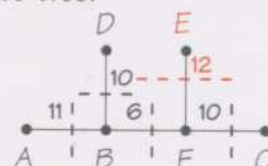


- max flow from  $E$  to  $F$  has value  $12$ ;

- min cut separates  $\{E\}, \{A, B, C, D, F\}$ ;

- new branch:  $\overset{12}{E} \dashv \overset{1}{F}$

- join branch to tree:





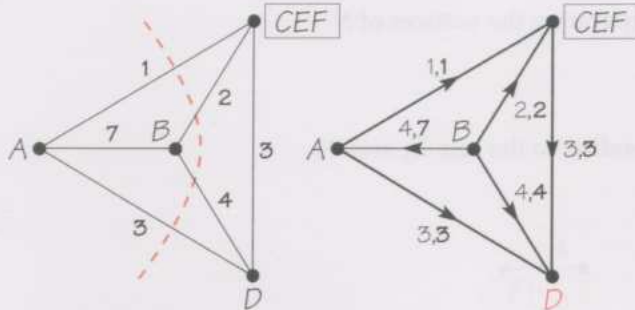


## 16 ANOTHER POSSIBLE SOLUTION TO PROBLEM 3

Third iteration

Split up vertex  $(BD)$ .

Condense vertices  $C, E$  and  $F$ .

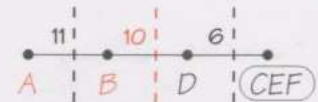


- max flow from  $B$  to  $D$  has value 10;

- min cut separates  $\{A, B\}, \{C, D, E, F\}$ ;

- new branch:  $B \xrightarrow{10} D$

- join branch to tree:

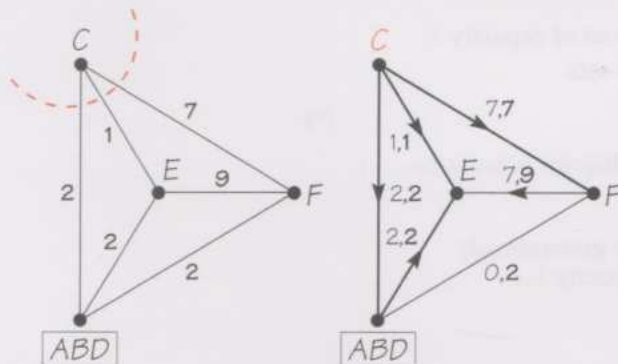


Fourth iteration

Split up vertex  $(CEF)$ .

Choose vertices  $C$  and  $E$ .

Condense vertices  $A, B$  and  $D$ .

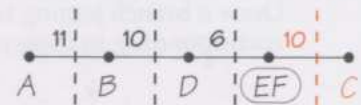


- max flow from  $C$  to  $E$  has value 10;

- min cut separates  $\{C\}, \{A, B, D, E, F\}$ ;

- new branch:  $C \xrightarrow{10} EF$

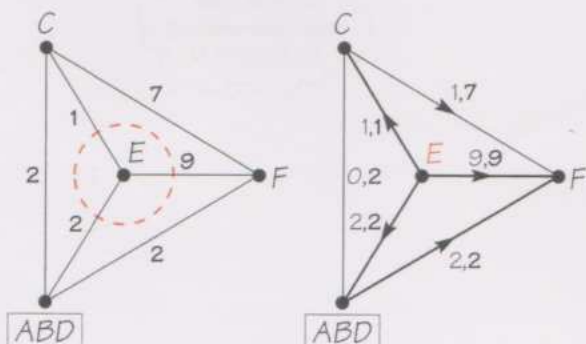
- join branch to tree:



Fifth iteration

Split up vertex  $(EF)$ .

Condense vertices  $A, B$  and  $D$ .

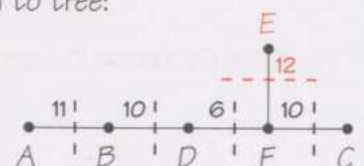


- max flow from  $E$  to  $F$  has value 12;

- min cut separates  $\{E\}, \{A, B, C, D, F\}$ ;

- new branch:  $E \xrightarrow{12} F$

- join branch to tree:



# Algorithm of Gomory and Hu for finding a cut tree

START with an undirected network  $N$  with  $n$  vertices.

First iteration

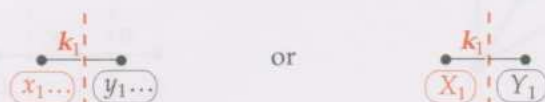
Choose any two vertices,  $x_1$  and  $y_1$  say, in  $N$ .

draw first branch

- In  $N$ , find a maximum flow from  $x_1$  to  $y_1$  of value  $k_1$ , say.
- In  $N$ , find a minimum cut of capacity  $k_1$  separating the vertices of  $N$  into two disjoint sets

$$X_1 = \{x_1, \dots\}, \quad Y_1 = \{y_1, \dots\}.$$

- Draw a branch with two vertices corresponding to the sets  $X_1$  and  $Y_1$ , and with capacity  $k_1$ :



At least one of the vertices,  $X_1$  say, will be a generalized vertex.

Second iteration

Choose any two vertices  $x_2$  and  $x_3$  in  $X_1$ .

Condense all the vertices in  $Y_1$  to a single vertex.

split up generalized vertex  $X_1$

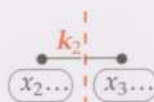
- In the condensed network, find a maximum flow from  $x_2$  to  $x_3$  of value  $k_2$ , say.
- In the condensed network, find a minimum cut of capacity  $k_2$  separating the vertices of  $N$  into two disjoint sets

$$X_2 = \{x_2, \dots\}, \quad Y_2 = \{x_3, \dots\}.$$

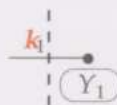
(\*)

- The cut (\*) splits the vertices of  $X_1$  into two disjoint subsets  $\{x_2, \dots\}$  and  $\{x_3, \dots\}$ .

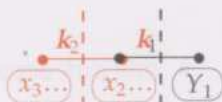
Draw a branch joining two vertices (possibly generalized) corresponding to these subsets, and with capacity  $k_2$ :



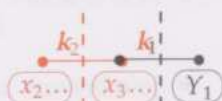
- Join the new branch to the existing part of the tree as follows:



at  $x_2...$  if  $x_2...$  belongs to the same part of the cut (\*) as  $Y_1$ :



at  $x_3...$  otherwise:



cuts in tree must correspond to cuts in  $N$

ONE OF THESE CASES MUST OCCUR.

### General iteration

Split up a generalized vertex as follows.

- Choose two vertices from any generalized vertex.
- Condense the vertices in the given network as appropriate.
- In the condensed network,
  - find a maximum flow from one of the chosen vertices to the other;
  - find a corresponding minimum cut;
  - draw the new branch of the cut tree;
  - add the new branch to the existing part of the cut tree.

see  
Frame 7

in the general case  
there are so  
many possibilities that  
precise instructions cannot  
be given here  
— a study of the examples  
shows the methods required

Note that for a general iteration, the situation becomes more complicated because

- it may be possible to condense more than one set of vertices;
- the new branch may have to be joined to a vertex which is not an end vertex.

see  
Frames 10  
and 13

see  
Frames 1  
and 13

After  $n$  iterations, STOP: a cut tree has been obtained.

### Points to note when applying the algorithm

- When splitting up a generalized vertex containing more than two vertices, it does not matter which two vertices you choose.
- There may be more than one way of achieving a maximum flow — choose any.
- For a particular maximum flow, there may be more than one minimum cut — choose any such that cuts *do not cross*.
- For a given undirected network, there may be more than one cut tree.

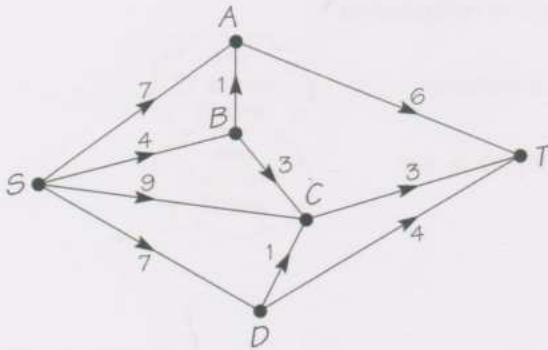
see  
Frames 5  
and 6

see  
Frame 16

## The shortest path algorithm

### 1 WORKED PROBLEM

Find the shortest path(s) from  $S$  to  $T$  in the given network.



USE

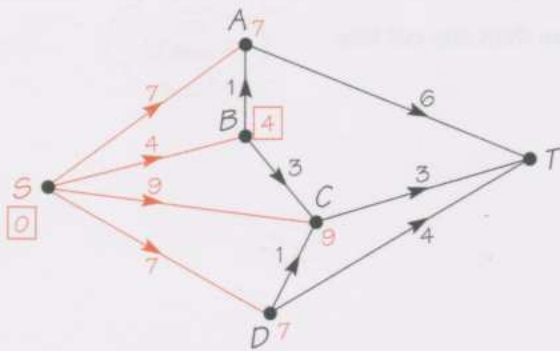
- labelling procedure
- tracing back procedure

### TRACK 2 & TRACK 3

### 2 SOLUTION TO WORKED PROBLEM

Start by giving  $S$  zero potential.

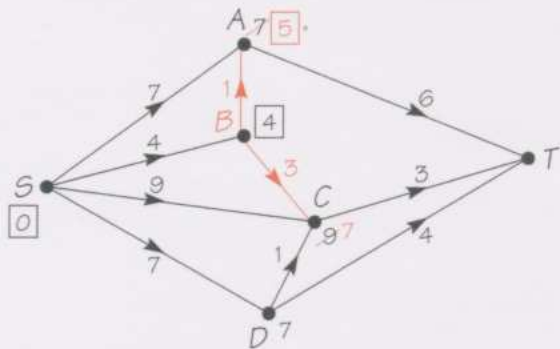
First iteration: label vertices reached directly from  $S$ .



iteration	origin vertex	vertices assigned labels				
		A	B	C	D	T
1	S	7	4	9	7	

$B$  has potential 4 assigned from  $S$ .

Second iteration: label vertices reached directly from  $B$ .



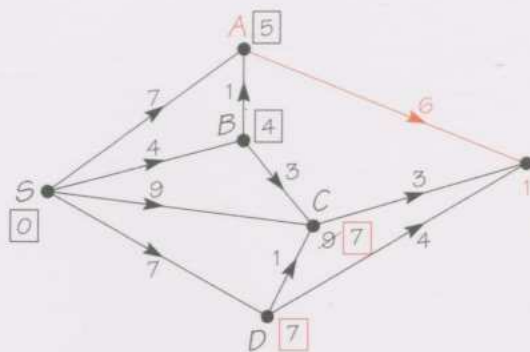
iteration	origin vertex	vertices assigned labels				
		A	B	C	D	T
1	S	7	4	9	7	
2	B	5		7		

$A$  has potential 5 assigned from  $B$ .



## 2 SOLUTION CONTINUED

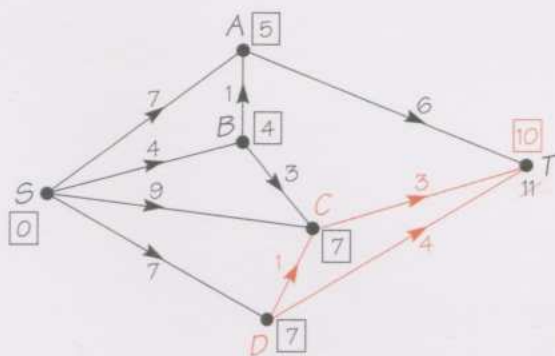
Third iteration: label vertices reached directly from **A**.



iteration	origin vertex	vertices assigned labels				
		A	B	C	D	T
1	S	<del>7</del>	<del>4</del>	<del>9</del>	<del>7</del>	<del>7</del>
2	B	<del>5</del>		<del>7</del>		
3	A			<del>7</del>	<del>7</del>	11

C has potential 7 assigned from B.  
D has potential 7 assigned from S.

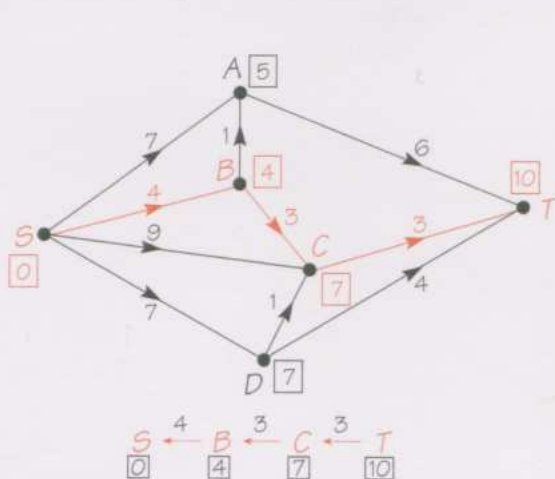
Fourth iteration: label vertices reached directly from **C** or **D**.



iteration	origin vertex	vertices assigned labels				
		A	B	C	D	T
1	S	<del>7</del>	<del>4</del>	<del>9</del>	<del>7</del>	<del>7</del>
2	B	<del>5</del>		<del>7</del>		
3	A			<del>7</del>	<del>7</del>	11
4	C					<del>10</del>
	D					11

T has potential 10 assigned from C.

The shortest path



iteration	origin vertex	vertices assigned labels				
		A	B	C	D	T
1	S	<del>7</del>	<del>4</del>	<del>9</del>	<del>7</del>	<del>7</del>
2	B	<del>5</del>		<del>7</del>		
3	A			<del>7</del>	<del>7</del>	11
4	C					<del>10</del>
	D					11

Shortest path is **SBCT** with length 10.

# 2. SOLUTION CONTINUED

Now suppose that nodes 1 and 2 are in  $S$ .



Now suppose that nodes 1, 2, and 3 are in  $S$ .



Now suppose that nodes 1, 2, and 3 are in  $S$ .



Now suppose that nodes 1, 2, and 3 are in  $S$ .

Node	1	2	3	4	5	6	7	8	9	10
Distance	0	0	1	2	3	4	5	6	7	8

Node	1	2	3	4	5	6	7	8	9	10
Distance	0	0	1	2	3	4	5	6	7	8

Node	1	2	3	4	5	6	7	8	9	10
Distance	0	0	1	2	3	4	5	6	7	8

Now suppose that nodes 1, 2, and 3 are in  $S$ .

### 3 THE SHORTEST PATH ALGORITHM: THE MAIN STEPS

**START** Assign potential 0 to  $S$ .

**GENERAL STEP** Consider the vertex (or vertices) just assigned a potential.

For each such vertex  $V$ , consider each vertex  $W$  which can be reached from  $V$  along an arc  $VW$ , and assign  $W$  the label

$$(\text{potential of } V) + (\text{distance } VW)$$

unless  $W$  already has a *smaller* label assigned from an earlier iteration.

When all such vertices  $W$  have been labelled, choose the smallest vertex label in the network which is not already a potential, and make it a potential at each vertex where it occurs.

**REPEAT** the GENERAL STEP with the new potential(s).

**STOP** when  $T$  has been assigned a potential.

The shortest distance from  $S$  to  $T$  is the potential of  $T$ .

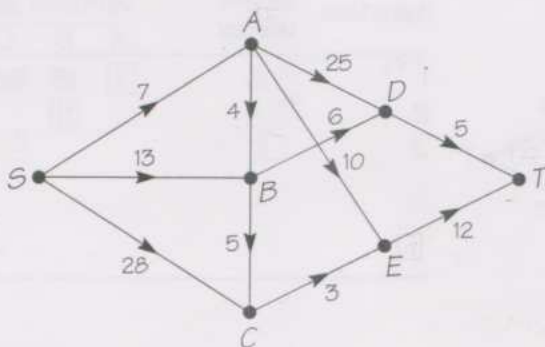
To find a shortest path, trace backwards from  $T$  and include an arc  $VW$  wherever

$$(\text{potential of } W) - (\text{potential of } V) = \text{distance } VW$$

until  $S$  is reached.

### 4 PROBLEM

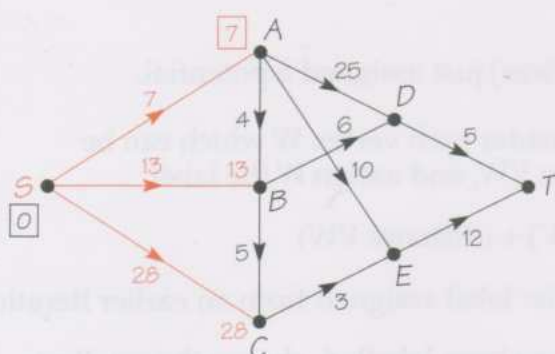
Find the shortest path from  $S$  to  $T$  in the following network.



## 5 SOLUTION TO PROBLEM

Start by giving  $S$  zero potential.

First iteration: label vertices reached directly from  $S$ .

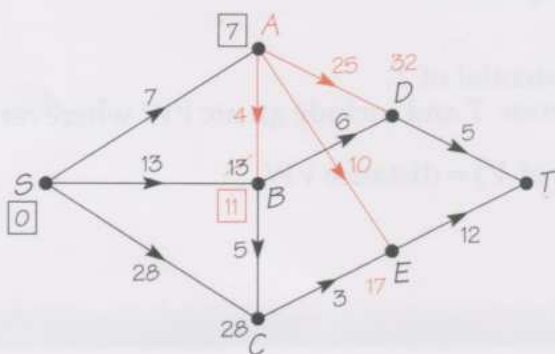


iteration	origin vertex	vertices assigned labels					
		A	B	C	D	E	T

1       $S$        $\boxed{7}$    13   28

$A$  has potential 7 assigned from  $S$ .

Second iteration: label vertices reached directly from  $A$ .



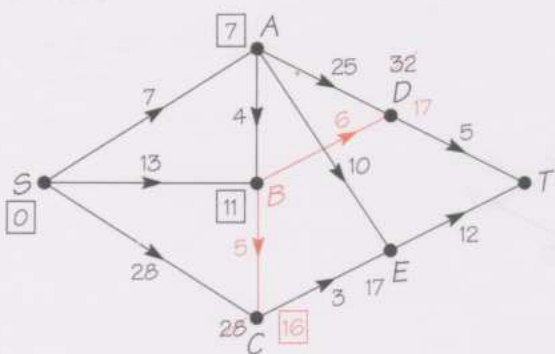
iteration	origin vertex	vertices assigned labels					
		A	B	C	D	E	T

1       $S$        $\boxed{7}$    13   28

2       $A$          $\boxed{11}$      32   17

$B$  has potential 11 assigned from  $A$ .

Third iteration: label vertices reached directly from  $B$ .



iteration	origin vertex	vertices assigned labels					
		A	B	C	D	E	T

1       $S$        $\boxed{7}$    13   28

2       $A$          $\boxed{11}$      32   17

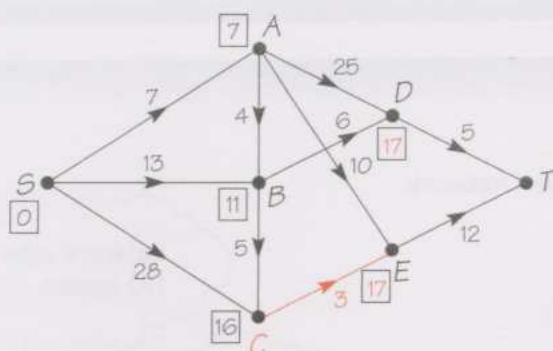
3       $B$            $\boxed{16}$    17

$C$  has potential 16 assigned from  $B$ .



## 5 SOLUTION CONTINUED

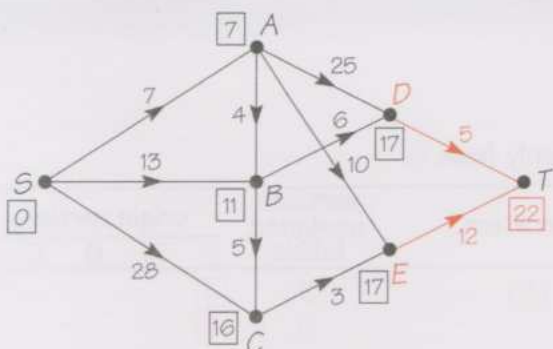
Fourth iteration: label vertices reached directly from *C*.



iteration	origin vertex	vertices assigned labels					
		A	B	C	D	E	T
1	S	7	13	28			
2	A		11		32	17	
3	B			16	17		
4	C				17	17	

*D* has potential 17 assigned from *B*;  
*E* has potential 17 assigned from *A*;

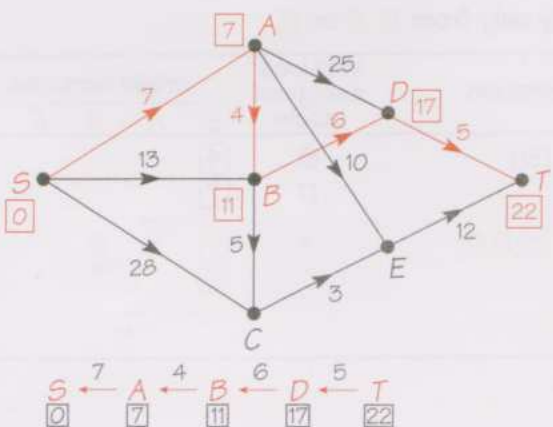
Fifth iteration: label vertices reached directly from *D* or *E*.



iteration	origin vertex	vertices assigned labels					
		A	B	C	D	E	T
1	S	7	13	28			
2	A		11		32	17	
3	B			16	17		
4	C				17	17	
5	D						22
	E						29

*T* has potential 22 assigned from *D*.

The shortest path



iteration	origin vertex	vertices assigned labels					
		A	B	C	D	E	T
1	S	7	13	28			
2	A		11		32	17	
3	B			16	17		
4	C				17	17	
5	D						22
	E						29

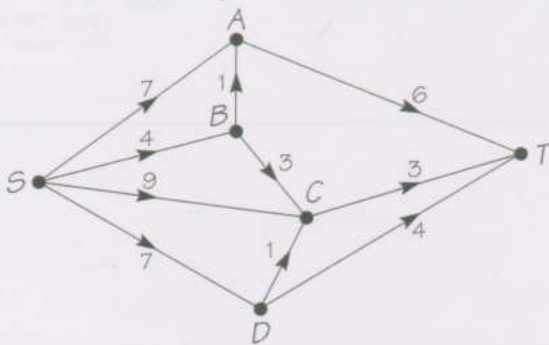
Shortest path is **SABDT** with length **22**.

# The longest path algorithm

TRACK 5

## 1 WORKED PROBLEM

Find the longest path(s) from  $S$  to  $T$  in the given network.



network with  
NO cycles

USE

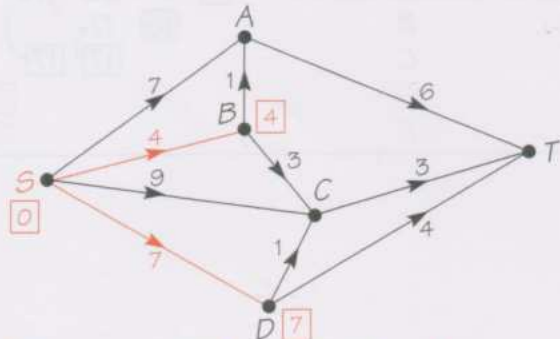
- labelling procedure
- tracing back procedure

TRACK 6 & TRACK 7

## 2 SOLUTION TO WORKED PROBLEM

Start by giving  $S$  zero potential.

First iteration: label vertices reached directly only from  $S$ .

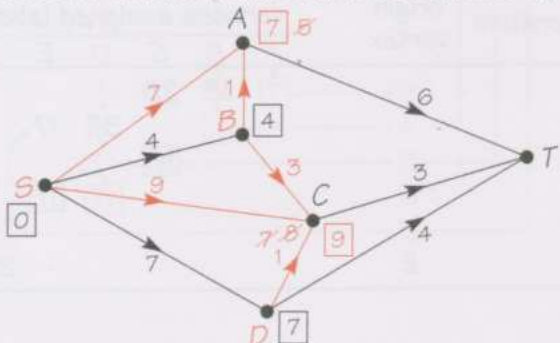


$B$  has potential 4 assigned from  $S$ .

$D$  has potential 7 assigned from  $S$ .

iteration	vertices assigned labels	origin vertices				
		$S$	$A$	$B$	$C$	$D$
1 ( $S$ )	$B$	4				
	$D$	7				

Second iteration: label vertices reached directly only from  $S$ ,  $B$  or  $D$ .



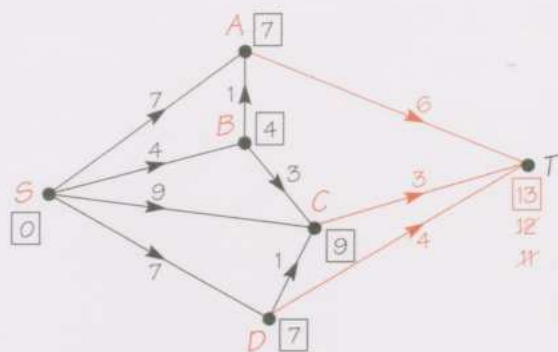
$A$  has potential 7 assigned from  $S$ .

$C$  has potential 9 assigned from  $S$ .

iteration	vertices assigned labels	origin vertices				
		$S$	$A$	$B$	$C$	$D$
1 ( $S$ )	$B$	4				
	$D$	7				
2 ( $S, B, D$ )	$A$	7		5		
	$C$	9		7		8

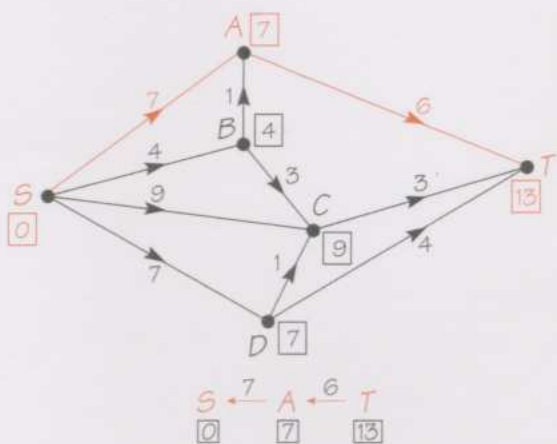
## 2 SOLUTION CONTINUED

Third iteration: label vertices reached directly only from  $S, A, B, C$  or  $D$ .



$T$  has potential 13 assigned from  $A$ .

The longest path



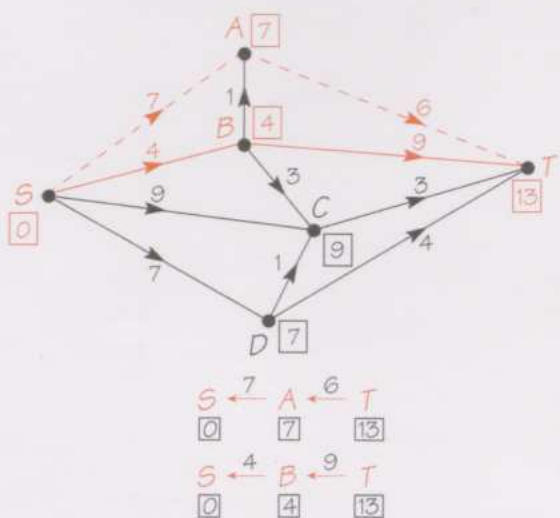
iteration	vertices assigned labels	origin vertices				
		$S$	$A$	$B$	$C$	$D$
1 ( $S$ )	$B$ $D$	$\boxed{4}$ $\boxed{7}$				
2 ( $S, B, D$ )	$A$ $C$	$\boxed{7}$ $\boxed{9}$		5 7		8
3 ( $S, A, B, C, D$ )	$T$		$\boxed{13}$		12	11

iteration	vertices assigned labels	origin vertices				
		$S$	$A$	$B$	$C$	$D$
1 ( $S$ )	$B$ $D$	$\boxed{4}$ $\boxed{7}$				
2 ( $S, B, D$ )	$A$ $C$	$\boxed{7}$ $\boxed{9}$		5 7		8
3 ( $S, A, B, C, D$ )	$T$		$\boxed{13}$		12	11

Longest path is  $SAT$  with length 13.

Postscript: two longest paths

Add arc  $BT$  with length 9:



iteration	vertices assigned labels	origin vertices				
		$S$	$A$	$B$	$C$	$D$
1 ( $S$ )	$B$ $D$	$\boxed{4}$ $\boxed{7}$				
2 ( $S, B, D$ )	$A$ $C$	$\boxed{7}$ $\boxed{9}$		5 7		8
3 ( $S, A, B, C, D$ )	$T$		$\boxed{13}$	$\boxed{13}$	12	11

Two longest paths:  $SAT$  and  $SBT$  with length 13.





### 3 THE LONGEST PATH ALGORITHM: THE MAIN STEPS

**START** Assign potential 0 to  $S$ .

**GENERAL STEP** Consider all vertices which can be reached directly only from vertices with known potentials.

For each such vertex  $W$ , consider each arc  $VW$  into  $W$ , and assign  $W$  the label

$$(\text{potential of } V) + (\text{distance } VW)$$

unless  $W$  already has a *larger* label.

When all such arcs  $VW$  have been considered, make the vertex label at  $W$  a potential.

**REPEAT** the GENERAL STEP with the new potential(s).

**STOP** when  $T$  has been assigned a potential.

The longest distance from  $S$  to  $T$  is the potential of  $T$ .

To find a longest path, work backwards from  $T$  and include an arc  $VW$  wherever

$$(\text{potential of } W) - (\text{potential of } V) = \text{distance } VW.$$

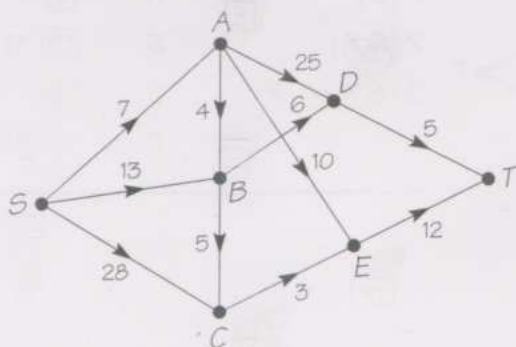
If the process does not terminate — for example, if the network has a cycle — then there is no longest path.

**REMEMBER!** — **shortest path algorithm:** look out from vertices of known potential;

**longest path algorithm:** look back to vertices of known potential.

### 4 PROBLEM

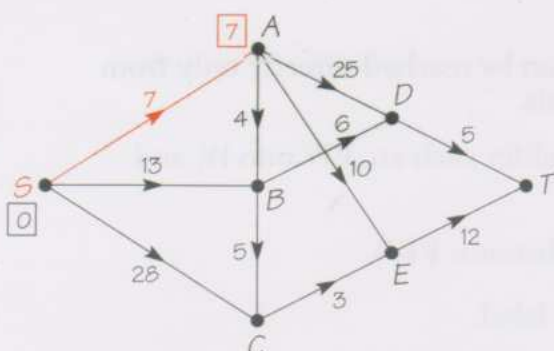
Find the longest path from  $S$  to  $T$  in the following network.



## 5 SOLUTION TO PROBLEM

Start by giving  $S$  zero potential.

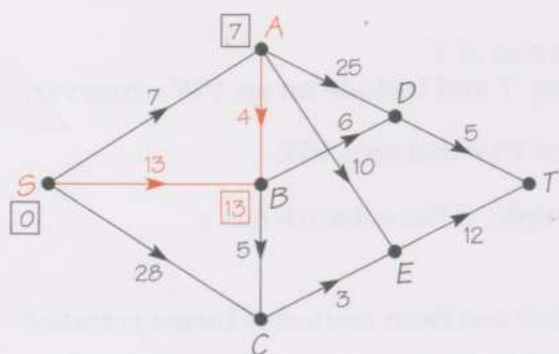
First iteration: label vertices reached directly only from  $S$ .



iteration	vertices assigned labels	origin vertices					
		$S$	A	B	C	D	E
1 ( $S$ )	A	7					

A has potential 7 assigned from  $S$ .

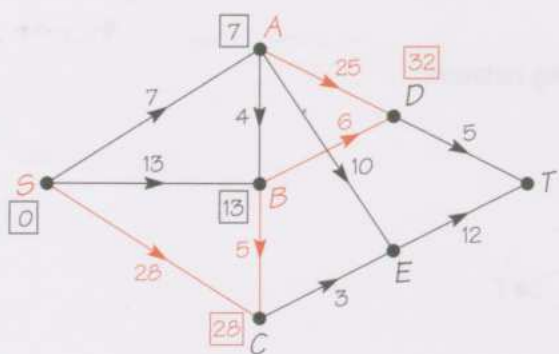
Second iteration: label vertices reached directly only from  $S$  or  $A$ .



iteration	vertices assigned labels	origin vertices					
		$S$	A	B	C	D	E
1 ( $S$ )	A	7					
2 ( $S, A$ )	B	13	11				

B has potential 13 assigned from  $A$ .

Third iteration: label vertices reached directly only from  $S$ ,  $A$  or  $B$ .

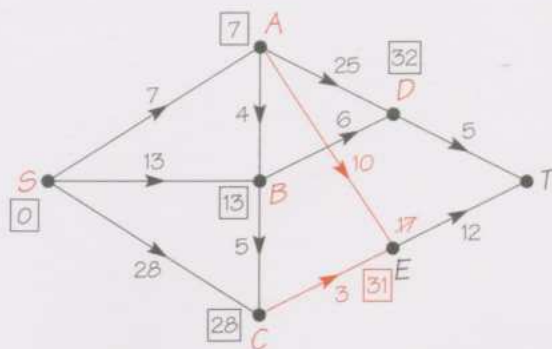


iteration	vertices assigned labels	origin vertices					
		$S$	A	B	C	D	E
1 ( $S$ )	A	7					
2 ( $S, A$ )	B	13	11				
3 ( $S, A, B$ )	C	28	18				
	D	32	19				

C has potential 28 assigned from  $S$ .  
D has potential 32 assigned from  $A$ .

## 5 SOLUTION CONTINUED

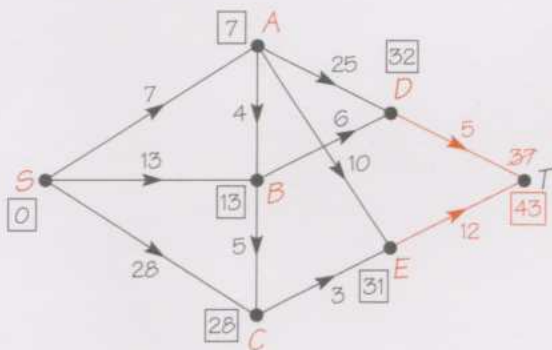
Fourth iteration: label vertices reached directly only from  $S$ ,  $A$ ,  $B$ ,  $C$  or  $D$ .



iteration	vertices assigned labels	origin vertices					
		S	A	B	C	D	E
1 ( $S$ )	A	<u>7</u>					
2 ( $S, A$ )	B	<u>13</u>	11				
3 ( $S, A, B$ )	C	<u>28</u>			18		
	D		<u>32</u>	19			
4 ( $S, A, B, C, D$ )	E		17		<u>31</u>		

$E$  has potential 31 assigned from  $C$ .

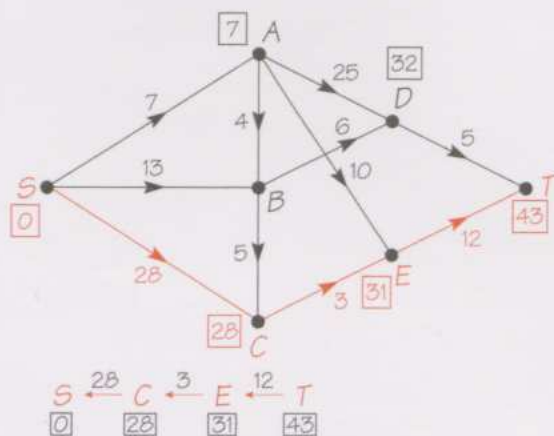
Fifth iteration: label vertices reached directly only from  $S$ ,  $A$ ,  $B$ ,  $C$ ,  $D$  or  $E$ .



iteration	vertices assigned labels	origin vertices					
		S	A	B	C	D	E
1 ( $S$ )	A	<u>7</u>					
2 ( $S, A$ )	B	<u>13</u>	11				
3 ( $S, A, B$ )	C	<u>28</u>			18		
	D		<u>32</u>	19			
4 ( $S, A, B, C, D$ )	E		17		<u>31</u>		
5 ( $S, A, B, C, D, E$ )	T						<u>37</u> <u>43</u>

$T$  has potential 43 assigned from  $E$ .

The longest path



iteration	vertices assigned labels	origin vertices					
		S	A	B	C	D	E
1 ( $S$ )	A	<u>7</u>					
2 ( $S, A$ )	B	<u>13</u>	11				
3 ( $S, A, B$ )	C	<u>28</u>			18		
	D		<u>32</u>	19			
4 ( $S, A, B, C, D$ )	E		17		<u>31</u>		
5 ( $S, A, B, C, D, E$ )	T						<u>37</u> <u>43</u>

Longest path is  $SCET$  with length 43.

For the first part of the question, the first part of the question is to find the value of  $x$  and  $y$ .

Angle	Reason	Value
$\angle A$	Angle in a straight line	$180^\circ - x$
$\angle B$	Angle in a straight line	$180^\circ - y$
$\angle C$	Angle in a straight line	$180^\circ - x$
$\angle D$	Angle in a straight line	$180^\circ - y$



Angle	Reason	Value
$\angle A$	Angle in a straight line	$180^\circ - x$
$\angle B$	Angle in a straight line	$180^\circ - y$
$\angle C$	Angle in a straight line	$180^\circ - x$
$\angle D$	Angle in a straight line	$180^\circ - y$



Angle	Reason	Value
$\angle A$	Angle in a straight line	$180^\circ - x$
$\angle B$	Angle in a straight line	$180^\circ - y$
$\angle C$	Angle in a straight line	$180^\circ - x$
$\angle D$	Angle in a straight line	$180^\circ - y$

